

CONCENTRATION-COMPACTNESS AT THE MOUNTAIN PASS LEVEL FOR NONLOCAL SCHRÖDINGER EQUATIONS

JOÃO MARCOS DO Ó AND DIEGO FERRAZ

ABSTRACT. The aim of this paper is to study a concentration-compactness principle for inhomogeneous fractional Sobolev space $H^s(\mathbb{R}^N)$ for $0 < s \leq N/2$. As an application we establish Palais-Smale compactness for the Lagrangian associated to the fractional Schrödinger equation $(-\Delta)^s u + a(x)u = f(x, u)$ for $0 < s < 1$. Moreover, we prove the existence of nontrivial nonnegative solutions to this class of elliptic equations for a wide class of possible singular potentials $a(x)$; not necessarily bounded away from zero. We consider possible oscillatory nonlinearities and that may not satisfy the Ambrosetti-Rabinowitz condition and for both cases; subcritical and critical growth range which are superlinear at origin.

CONTENTS

1. Introduction	2
1.1. Outline	4
2. Profile Decomposition for weak convergence in fractional Sobolev spaces	4
3. Nonlinear fractional Schrödinger equation	6
3.1. Hypothesis	6
3.2. Statement of the main existence results	9
4. Preliminaries	14
4.1. Fractional Sobolev spaces	14
4.2. The s -harmonic extension	16
4.3. D-weak convergence and dislocation spaces	18
5. Proof of Theorem 2.2	19
6. Variational settings	20
6.1. Behavior of weak decomposition convergence under nonlinearities	26
6.2. Pohozaev Identity	30
7. Proof of Theorem 3.1	34
8. Proof of Theorem 3.2	36
9. Proof of Theorem 3.3	38
10. Proof of Theorem 3.4	42
References	45

2000 *Mathematics Subject Classification.* 35P15, 35P30, 35R11.

Key words and phrases. Fractional Schrödinger equation, Fractional Laplacian, Concentration-compactness.

1. INTRODUCTION

The main goal of the present work is to analyze concentration-compactness principles for inhomogeneous fractional Sobolev spaces. As an application we address questions on compactness for the associated energy functional to the following nonlocal Schrödinger equation

$$(-\Delta)^s u + a(x)u = f(x, u) \quad \text{in } \mathbb{R}^N, \quad (\mathcal{P}_s)$$

where $0 < s < 1$, and $(-\Delta)^s$ is the fractional Laplacian defined by the relation

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}u(\xi), \quad \xi \in \mathbb{R}^N,$$

where $\mathcal{F}u$ is the Fourier transform of u , i.e.

$$\mathcal{F}u(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(\xi) e^{-i\xi \cdot x} d\xi, \quad x \in \mathbb{R}^N. \quad (1.1)$$

Let \mathcal{S} be the Schwartz space consisting of rapidly decaying C^∞ functions in \mathbb{R}^N which, together with all their derivatives, vanish at the infinity faster than any power of $|x|$. Equivalently, if $u \in \mathcal{S}$ the fractional Laplacian of u can be computed by the following singular integral

$$(-\Delta)^s u(x) = C(N, s) \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(0)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

for a suitable positive normalizing constant

$$C(N, s) = \left(\int_{\mathbb{R}^N} \frac{1 - \cos \varsigma_1}{|\varsigma|^{N+2s}} d\varsigma \right)^{-1}. \quad (1.2)$$

We refer to [18, 35, 55] for an introduction to the fractional Laplacian operator.

During the past years there has been a considerable amount of research on nonlinear elliptic problems involving fractional Laplacian motivated from the fact this class of problems arise naturally in several branches of mathematical physics. For instance, solutions of (\mathcal{P}_s) , can be seen as stationary states (corresponding to solitary waves) of nonlinear Schrödinger equations of the form $i\phi_t - (-\Delta)^s \phi + a(x)\phi + f(x, \phi) = 0$ in \mathbb{R}^N . For more motivation we refer to [2, 3, 8], where the reader can see that equations involving the operator $(-\Delta)^s$ arises from several areas of science such as biology, chemistry or finance.

This paper is motivated by recent advances in the study of existence of solutions for nonlinear and nonlocal Schrödinger field equations. First, we would like to mention the progress involving potentials $a(x)$ bounded away from zero and nonlinearities with *subcritical growth*. In [52] S. Secchi investigated the existence of ground state solutions for fractional Schrödinger equations by using a minimization argument on the Nehari manifold. He proved existence results under suitable assumptions on the behavior of the potential $a(x)$ and superlinear growth conditions on the nonlinearity. See also [29], where B. Feng proved the existence of ground state solutions of (\mathcal{P}_s) , for the particular case $f(x, t) = |t|^{p-2}t$, $2 < p < 2(N + 2s)/N$, $N \geq 2$, by using the P.-L. Lions concentration-compactness principle (see [40]). R. Lehrer et al. [36] studied the existence of solutions through projection over an appropriated Pohozaev manifold, assuming that $f(x, t) = a(x)f_0(t)$, where $f_0(t)$ is asymptotically linear that is, $\lim_{t \rightarrow \infty} f_0(t)/t = 1$ and $\lim_{|x| \rightarrow \infty} a(x) = a_\infty > 0$. For the local case ($s = 1$), R. de Marchi [15] studied existence of nontrivial solutions for (\mathcal{P}_s) assuming that $a(x)$ and $f(x, t)$ are asymptotically 1-periodic in each x_i , $i = 1, \dots, N$, combining variational methods and the concentration-compactness principle, and also proved existence of ground state solutions when $a(x)$ and $f(x, t)$ are 1-periodic in each x_i , $i = 1, \dots, N$, without assuming that $t \mapsto f(x, t)t^{-1}$ is an increasing function. By using similar approach, H. Zhang et al [68], studied existence of ground state and infinitely many

geometrically distinct solutions for Eq. (\mathcal{P}_s) , based on the method of Nehari manifold and Lusternik-Schnirelmann category theory. Moreover, for recent works on nonlinear Schrödinger equations where the classical Ambrosetti-Rabinowitz condition is not required we cite [15, 36, 68].

Problems involving potentials bounded away from zero and *critical Sobolev exponent*, precisely, when $f(x, t) = g(x, t) + |t|^{2_s^*-2}t$, where $g(x, t)$ have subcritical growth, we may refer to [37, 53, 54] and the references given there. In these works, it was crucial the presence of perturbation $g(x, t)$ of the critical power $|t|^{2_s^*-2}t$. Moreover, it was assumed the following condition on the potential $0 < \inf_{x \in \mathbb{R}^N} a(x) < \liminf_{|x| \rightarrow \infty} a(x)$ which was introduced by P.L. Rabinowitz in [47] to study the local case of Eq. (\mathcal{P}_s) (see also for the critical case [43]). We cite [13, 17, 57] for works on local Schrödinger equations with nonlinearities of the pure critical power type (without subcritical perturbation term) and inverse square type potentials. For the fractional case we cite [20], where it was studied the existence qualitative properties of positive solutions.

Motivated by the above works, we obtain existence of nontrivial solutions for Eq. (\mathcal{P}_s) in several cases, which were not considered by the aforementioned papers. Our potential $a(x)$ may change sign, can have singular points of blow up and even vanish, and the nonlinearity can be considered with critical or subcritical oscillatory growth. To prove some of these existence results it is crucial for our argument to analyze the Palais-Smale compactness of the associated functional at the mountain pass level, i.e. to prove that these sequences are relatively compact.

In the subcritical case we assume a condition on the potential $a(x)$ which ensures the continuous embedding of the associated space of functions similar to [56]. Nevertheless differently from [56], we do not impose assumption on $a(x)$ to guarantee the compactness of the Sobolev embedding. To compensate, we ask that the limit of $a(x)$, as $|x|$ goes to infinity, exists and is positive, or alternatively, that $a(x)$ is 1-periodic in each x_i , $i = 1, \dots, N$. This leads to separate the study in two cases in terms of potentials, in the same way as in [15]. Moreover, by considering similar assumptions made in [16], the potential does not need to be bounded from below by a constant. We also take account the case where the nonlinearity has oscillatory behavior and does not satisfies the typical assumption of Ambrosetti-Rabinowitz. Similar to [15], the nonlinearity $f(x, t)$ is supposed to has a periodic asymptote $f_{\mathcal{P}}(x, t)$, which allow us to “transfer” the usual assumptions to it. Also we mention that we complement and improve some results in [15], since we consider the fractional case and we do not require the monotonicity of $t \mapsto f_{\mathcal{P}}(x, t)t^{-1}$.

In the critical case, inspired in some ideas contained in [13], we treated in this work a class of potentials somehow different, since we consider a general class that include as a particular case the inverse fractional square potential $a(x) = -\lambda|x|^{-2s}$, where $0 < \lambda < \Lambda_{N,s}$ and $\Lambda_{N,s}$ is the sharp constant of the Hardy-Sobolev inequality,

$$\Lambda_{N,s} \int_{\mathbb{R}^N} |x|^{-2s} u^2 dx \leq \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u|^2 d\xi, \quad \forall u \in C_0^\infty(\mathbb{R}^N), \quad (1.3)$$

where $\Lambda_{N,s}$ is the sharp constant of this inequality given by

$$\Lambda_{N,s} := 2^{2s} \frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}, \quad 0 < s < 1, \quad N > 2s, \quad (1.4)$$

and Γ is the well known Gamma function. Further details about (1.3) can be found in [31] and [67]. Here we consider self-similar nonlinearities which generalize the idea of oscillations about the critical power $|t|^{2_s^*-2}t$, turning the approach by variational methods more involved. This class of functions were introduced in [22] and for the local case in [51, 63–65].

We are able to avoid the assumption of the monotonicity of $t \mapsto f_{\mathcal{P}}(x, t)t^{-1}$ by comparing the minimax level of the associated energy functional of Eq. (\mathcal{P}_s) with the one of the associated

limit problem. To this end we use a Pohozaev type identity and an appropriated concentration-compactness principle. The proof of this identity is essentially based in the use of the so called s -harmonic extension introduced by L. Caffarelli and L. Silvestre [9] and remarks contained in [24] and [33], which allow us to “transform” the nonlocal problem (\mathcal{P}_s) in a local one, from this we may apply a truncation argument. Our method of proof is more general than the usual one, in the sense that in our argument we do not have to study the behavior of solutions in the whole space \mathbb{R}^N ; and we also can consider singular potentials (see Proposition 6.11).

It is worth to mention that the main difficulty to approach problem (\mathcal{P}_s) using variational methods lies on the lack of compactness, which roughly speaking, originates from the invariance of \mathbb{R}^N with respect to translation and dilation and, analytically, appears because of the non-compactness of the Sobolev embedding. We are able to overcome this difficulty by relying in a concentration-compactness principle by means of profile decomposition for weak convergence in inhomogeneous fractional Sobolev spaces, which can be considered as extensions of the Banach-Alaoglu theorem (see Theorem 2.2). This kind of results were considered in various settings, for instance we may cite the ones in [30, 32, 45, 58, 59]. It describes how the convergence of a bounded sequence fails in the considered space. Our approach in this matter was motivated by [14] and based in the abstract version of profile decomposition in Hilbert spaces due to K. Tintarev and K.-H. Fieseler [65]. It seems for us that this approach is more appropriated to study existence of nontrivial solutions for problems like (\mathcal{P}_s) , under our settings, then the usuals ones using P.-L. Lions concentration-compactness principle (see [28, Lemma 2.2]).

Another important issue of this paper is the study of the existence of ground state solutions for (\mathcal{P}_s) , i.e., nontrivial solutions with least possible energy. We prove the existence of ground states in three cases: First when (\mathcal{P}_s) is invariant under the action of translations in \mathbb{Z}^N (subcritical growth), second when (\mathcal{P}_s) is invariant under dilations $\gamma^{(N-2s)j/2}u(\gamma\cdot)$ (critical growth), and third when the monotonicity of $t \mapsto f(x, t)t^{-1}$ is considered.

1.1. Outline. The paper is organized as follows. Sect. 2 is devoted to the description of the profile decomposition of bounded sequence in the fractional Sobolev space, which are often used in our work. In Sect. 3, we give some applications of the profile decomposition to study the existence of mountain-pass type solutions of (\mathcal{P}_s) , for autonomous and non-autonomous case. In Sect. 4, we state some basic results (without prove) on the fractional Sobolev spaces. In Sect. 5, by establishing how translation acts in the inhomogeneous fractional Sobolev spaces, we prove the abstract result stated in Sect. 2. In Sect. 6, we provide a suitable variational settings to prove our main results, more precisely, we describe the limit under the profile decomposition of the Palais-Smale sequence at the mountain pass level of the Lagrangian of (\mathcal{P}_s) and we prove that solutions for (\mathcal{P}_s) in the autonomous case satisfies a Pohozaev type identity. Sections 7, 8, 9 and 10 are dedicated to the proof of our main results concerning existence of mountain pass solutions for Eq. (\mathcal{P}_s) .

2. PROFILE DECOMPOSITION FOR WEAK CONVERGENCE IN FRACTIONAL SOBOLEV SPACES

Assume $0 < s < N/2$ and let $\mathcal{D}^{s,2}(\mathbb{R}^N)$ be the homogeneous fractional Sobolev space, which are defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$[u]^2 := \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u|^2 \, d\xi.$$

It is well known that $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is continuous embedded in the space $L^{2_s^*}(\mathbb{R}^N)$, where $2_s^* = 2N/(N - 2s)$ is the critical Sobolev exponent. The following results represents the theory

developed by K. Tintarev and K.-H. Fieseler in [65] for the context of fractional Sobolev spaces, and was studied in [22] and [45].

Theorem A. [22, Theorem 2.1] *Let $(u_k) \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a bounded sequence, $\gamma > 1$, $0 < s < 1$ and $N > 2s$. Then there exists $\mathbb{N}_* \subset \mathbb{N}$, disjoint sets (if non-empty) $\mathbb{N}_0, \mathbb{N}_-, \mathbb{N}_+ \subset \mathbb{N}$, with $\mathbb{N}_* = \mathbb{N}_0 \cup \mathbb{N}_+ \cup \mathbb{N}_-$ and sequences $(w^{(n)})_{n \in \mathbb{N}_*}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $(y_k^{(n)})_{k \in \mathbb{N}}$ in \mathbb{Z}^N , $(j_k^{(n)})_{k \in \mathbb{N}}$ in \mathbb{Z} , $n \in \mathbb{N}_*$, such that, for a subsequence sequence of (u_k) ,*

$$\gamma^{-\frac{N-2s}{2}j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}} \cdot + y_k^{(n)}) \rightharpoonup w^{(n)} \text{ as } k \rightarrow \infty, \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N), \quad (2.1)$$

$$|j_k^{(n)} - j_k^{(m)}| + |\gamma^{j_k^{(n)}}(y_k^{(n)} - y_k^{(m)})| \rightarrow \infty, \text{ as } k \rightarrow \infty, \text{ for } m \neq n, \quad (2.2)$$

$$\sum_{n \in \mathbb{N}_*} [w^{(n)}]_s^2 \leq \limsup_k [u_k]_s^2, \quad (2.3)$$

$$u_k - \sum_{n \in \mathbb{N}_*} \gamma^{\frac{N-2s}{2}j_k^{(n)}} w^{(n)}(\gamma^{j_k^{(n)}}(\cdot - y_k^{(n)})) \rightarrow 0, \text{ as } k \rightarrow \infty, \text{ in } L^{2^*}_s(\mathbb{R}^N), \quad (2.4)$$

and the series in (2.4) converges uniformly in k . Furthermore, $1 \in \mathbb{N}_0$, $y_k^{(1)} = 0$; $j_k^{(n)} = 0$ whenever $n \in \mathbb{N}_0$; $j_k^{(n)} \rightarrow -\infty$ whenever $n \in \mathbb{N}_-$; and $j_k^{(n)} \rightarrow +\infty$ whenever $n \in \mathbb{N}_+$.

It is worth to mention that Theorem A can be used to prove the fractional version of Lions concentration-compactness principle proved in [45, Theorem 5]. Indeed, Theorem A improves [45, Theorem 5] for the case $\Omega = \mathbb{R}^N$, since the sums of Dirac masses that appears in this result comes from the profiles given in (2.4). Also, the new notion of criticality introduced in [22] together with the concentration-compactness given in the profile decomposition Theorem A can lead to a new way to approach elliptic problems involving nonlinearities with critical growth and the fractional Laplacian, for instance, replacing the well known nonlinearity $f(x, t) = K(x)|t|^{2^*_s-2}t$, which is often considered to studied existence results for Eq. (\mathcal{P}_s) with aid of [45, Theorem 5], for a general self-similar function (for more details see [22, Sect. 3.1]).

Remark 2.1. One can consider the closed subspace of $\mathcal{D}^{s,2}(\mathbb{R}^N)$ consisting of radial functions, that is,

$$\mathcal{D}^{s,2}_{\text{rad}}(\mathbb{R}^N) = \{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) : u(x) = u(y), \text{ provided that } |x| = |y|\},$$

to obtain more compactness. Precisely, we have that $w^{(n)}$ belongs to $\mathcal{D}^{s,2}_{\text{rad}}(\mathbb{R}^N)$ with $w^{(n)} = 0$, for all $n \in \mathbb{N}_0$. The proof of this fact follows the same arguments as in [65, Proposition 5.1].

As expected Theorem A is applied to study (\mathcal{P}_s) when $f(x, t)$ has critical growth.

In this paper, we prove the inhomogeneous case of Theorem A, that is, for the spaces $H^s(\mathbb{R}^N)$, which are defined as

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\cdot|^{2s} \mathcal{F}u \in L^2(\mathbb{R}^N)\}, \quad 0 < s \leq N/2,$$

with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u|^2 + u^2 \, d\xi.$$

It is known that $H^s(\mathbb{R}^N)$ is continuous embedded in $L^p(\mathbb{R}^N)$, for $2 \leq p \leq 2^*_s$, in the case where $N > 2s$, and in $L^p(\mathbb{R}^N)$, for $2 \leq p < \infty$, in the case where $N = 2s$. The following version of Theorem A will be used to study about the existence of solutions for the case where $f(x, t)$ has subcritical growth in (\mathcal{P}_s) and is stated next.

Theorem 2.2. *Let $(u_k) \subset H^s(\mathbb{R}^N)$ be a bounded sequence with $0 < s \leq N/2$. Then there exists $\mathbb{N}_0 \subset \mathbb{N}$, and sequences $(w^{(n)})_{n \in \mathbb{N}_0}$ in $H^s(\mathbb{R}^N)$, $(y_k^{(n)})_{k \in \mathbb{N}}$ in \mathbb{Z}^N , $n \in \mathbb{N}_0$, such that, up to subsequence of (u_k)*

$$u_k(\cdot + y_k^{(n)}) \rightharpoonup w^{(n)}, \text{ as } k \rightarrow \infty, \text{ in } H^s(\mathbb{R}^N), \quad (2.5)$$

$$|y_k^{(n)} - y_k^{(m)}| \rightarrow \infty, \text{ as } k \rightarrow \infty, \text{ for } m \neq n, \quad (2.6)$$

$$\sum_{n \in \mathbb{N}_0} \|w^{(n)}\|^2 \leq \limsup_k \|u_k\|^2, \quad (2.7)$$

$$u_k - \sum_{n \in \mathbb{N}_0} w^{(n)}(\cdot + y_k^{(n)}) \rightarrow 0, \text{ as } k \rightarrow \infty, \text{ in } L^p(\mathbb{R}^N), \quad (2.8)$$

for any $p \in (2, 2_s^*)$. Moreover, $1 \in \mathbb{N}_0$, $y_k^{(1)} = 0$ and the series in (2.8) converges uniformly in k .

Remark 2.3. These profile decompositions for bounded sequence are unique up to a permutation of index, and up to constant operator. See [65, Proposition 3.4].

As it can be seen, Theorem 2.2 describes how bounded sequences in $H^s(\mathbb{R}^N)$ fails to converges in $L^p(\mathbb{R}^N)$, $2 < p < 2_s^*$. This “error” of convergence is generated by the invariance of action of translations in $H^s(\mathbb{R}^N)$. As one can see, Theorem 2.2 is the fractional counterpart of [65, Corollary 3.3], which is possible thanks to the results made in [14]. Moreover, we mention that Theorem 2.2 is an alternative to the well known fractional Lions Lemma of compactness (see [28, Lemma 2.2]), as can be seen in Sect. 7.

3. NONLINEAR FRACTIONAL SCHRÖDINGER EQUATION

3.1. Hypothesis. In order to describe our results on the energy functional of (\mathcal{P}_s) in a more precisely way, next we state the main assumptions on the potential $a(x)$ and the nonlinearity $f(x, t)$ respectively. We always assume that $N > 2s$ and $0 < s < 1$. Also, in what follows we denote $\|\cdot\|_p$ and $\|\cdot\|_\infty$ the norms of the spaces $L^p(\mathbb{R}^N)$, $1 \leq p < \infty$ and $L^\infty(\mathbb{R}^N)$ respectively, moreover $|A|$ denotes the Lebesgue measure of the set $A \subset \mathbb{R}^N$.

3.1.1. Subcritical case.

- Assumptions on $a(x) = V(x) - b(x)$.

(V₁) $V(x) \in L_{\text{loc}}^\sigma(\mathbb{R}^N)$ for some $\sigma > 2N/(N + 2s)$. $V(x)$ is 1 – periodic in x_i , $i = 1, \dots, N$.

$$(V_2) \quad \begin{cases} \mathcal{C}_V := \inf_{u \in C_0^\infty(\mathbb{R}^N), \|u\|_2=1} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + V(x)u^2 dx > 0, \\ \exists \mathcal{B} > 0 \text{ such that (s.t.) } V(x) \geq -\mathcal{B} \text{ almost everywhere (a.e.) } x \in \mathbb{R}^N \end{cases}$$

$$(V_3) \quad \begin{cases} 0 \leq b(x) \in L^\beta(\mathbb{R}^N), \text{ for some } \beta > N/2s. \|b(x)\|_\beta < \mathcal{C}_V^{(\beta)}, \text{ where} \\ \mathcal{C}_V^{(\beta)} := \inf_{u \in H_V^s(\mathbb{R}^N), \|u\|_{2\beta'}=1} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + V(x)u^2 dx, \quad \beta' = \beta/(\beta - 1). \end{cases}$$

(V₄) $V(x) \in L_{\text{loc}}^\sigma(\mathbb{R}^N)$, for some $\sigma > N/2s$. $\exists V_\infty := \lim_{|x| \rightarrow \infty} V(x) > 0$.

- Assumptions on $f(x, t)$.

$$(f_1) \quad \begin{cases} f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R} \text{ is a Carathéodory function satisfying the growth condition} \\ \forall \varepsilon > 0, \exists C_\varepsilon > 0, p_\varepsilon \in (2, 2_s^*) \text{ s.t.} \\ |f(x, t)| \leq \varepsilon(|t| + |t|^{2_s^*-1}) + C_\varepsilon |t|^{p_\varepsilon-1}, \text{ a.e. } x \in \mathbb{R}^N, \forall t \in \mathbb{R}. \end{cases}$$

$$(f_2) \quad \exists \mu > 2 \quad \text{s.t.} \quad \mu F(x, t) := \mu \int_0^t f(x, \tau) d\tau \leq f(x, t)t, \text{ a.e. } x \in \mathbb{R}^N, \forall t \in \mathbb{R}$$

$$(f_3) \quad \begin{cases} \exists R > 0, t_0 > 0, x_0 \in \mathbb{R}^N \text{ s.t.} \\ |B_R| \inf_{B_R(x_0)} F(x, t_0) + |B_{R+1} \setminus B_R| \inf_{(x,t) \in (B_{R+1}(x_0) \setminus B_R(x_0)) \times [0, t_0]} F(x, t) > 0, \end{cases}$$

$$(f_4) \quad \begin{cases} \lim_{t \rightarrow 0} \frac{f(x, t)}{t} = 0 \quad \text{and} \quad \lim_{|t| \rightarrow \infty} \frac{F(x, t)}{t^2} = \infty, \text{ uniformly in } x. \\ \forall \text{ compact set } K \subset \mathbb{R}, \exists C = C(K) > 0 \text{ s.t. } |f(x, t)| \leq C, \text{ a.e. } x \in \mathbb{R}^N, \forall t \in K. \end{cases}$$

$$(f_5) \quad \forall 0 < a < b, \inf_{x \in \mathbb{R}^N} \inf_{a \leq |t| \leq b} \mathcal{F}(x, t) > 0, \text{ where } \mathcal{F}(x, t) := \frac{1}{2} f(x, t)t - F(x, t).$$

$$(f_6) \quad \begin{cases} \exists p_0 > \max\{1, N/2s\}, a_0, R_0 > 0 \text{ s.t.} \\ |f(x, t)|^{p_0} \leq a_0 |t|^{p_0} \mathcal{F}(x, t), \text{ a.e. } x \in \mathbb{R}^N, \forall |t| > R_0. \end{cases}$$

$$(f_7) \quad \begin{cases} \exists 1\text{-periodic function } f_{\mathcal{P}}(x, t) \text{ in each } x_i, i = 1, \dots, N, \text{ s.t.} \\ \lim_{|x| \rightarrow \infty} |f(x, t) - f_{\mathcal{P}}(x, t)| = 0, \text{ uniformly in compact sets of } \mathbb{R}. \\ f_{\mathcal{P}}(x, t) \text{ satisfies } (f_1) \text{ and either } (f_2)-(f_3) \text{ or } (f_4). \end{cases}$$

$$(f_8) \quad \text{For a.e. } x \in \mathbb{R}^N, \text{ the function } t \mapsto \frac{f_{\mathcal{P}}(x, t)}{|t|}, \text{ is strict increasing in } \mathbb{R}.$$

In the next condition we are assuming that $f_{\mathcal{P}}(x, t)$ in (f_7) is independent of t and we denote $f_\infty(t) = f_{\mathcal{P}}(t)$.

$$(f_9) \quad f_\infty(t) \in C^1(\mathbb{R}), \exists t_0 > 0 \text{ s.t. } F_\infty(t_0) - \frac{V_\infty}{2} t_0^2 > 0, \text{ where } F_\infty(t) = \int_0^t f_\infty(\tau) d\tau.$$

We look for solutions in the Hilbert space $H_V^s(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_V^2 := \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + V(x) u^2 dx \quad \text{and} \quad (u, v)_V := \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v + V(x) uv dx,$$

see Proposition 6.1. Writing, $a(x) = V(x) - b(x)$, and assuming (V_3) and (f_1) we can see that the functional associated with (\mathcal{P}_s) , $I : H_V^s(\mathbb{R}^N) \rightarrow \mathbb{R}$ given by

$$I(u) = \frac{1}{2} \|u\|_V^2 - \frac{1}{2} \int_{\mathbb{R}^N} b(x) u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx, \quad (3.1)$$

is well defined, belongs to $C^1(H_V^s(\mathbb{R}^N))$ (see [46]), also

$$I'(u) \cdot v = \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v + (V(x) - b(x)) uv dx - \int_{\mathbb{R}^N} f(x, u) v dx, \quad u, v \in H_V^{s,2}(\mathbb{R}^N).$$

Thus critical points of I correspond to weak solutions of (\mathcal{P}_s) and conversely. Consider the minimax level

$$c(I) = \inf_{\gamma \in \Gamma_I} \sup_{t \geq 0} I(\gamma(t)), \quad (3.2)$$

where

$$\Gamma_I = \left\{ \gamma \in C([0, \infty), H_V^s(\mathbb{R}^N)) : \gamma(0) = 0, \lim_{t \rightarrow \infty} I(\gamma(t)) = -\infty \right\}. \quad (3.3)$$

Associated with the limits given in (V_4) , (f_7) , (f_9) , we consider the C^1 functionals

$$\begin{aligned} I_{\mathcal{P}}(u) &:= \frac{1}{2} \|u\|_V^2 - \int_{\mathbb{R}^N} F_{\mathcal{P}}(x, u) \, dx, \quad u \in H_V^s(\mathbb{R}^N), \\ I_{\infty}(u) &:= \frac{1}{2} \|u\|_{V_{\infty}}^2 - \int_{\mathbb{R}^N} F_{\infty}(u) \, dx, \quad u \in H_V^s(\mathbb{R}^N), \end{aligned}$$

where $F_{\mathcal{P}}(x, t) = \int_0^t f(x, \tau) d\tau$. Similarly, as in (3.2) and (3.3) , we can define $c(I_{\mathcal{P}})$, $c(I_{\infty})$, $\Gamma_{I_{\mathcal{P}}}$ and $\Gamma_{I_{\infty}}$. Next we state the assumption on the minimax levels to guarantees compactness of the Palais-Smale sequences at the mountain pass level of I .

$$(f_{10}) \quad c(I) < c(I_{\mathcal{P}});$$

$$(f'_{10}) \quad c(I) < c(I_{\infty});$$

In the autonomous case, where $f(x, t) = f(t)$, we consider the following variant of (f_3) .

$$(f'_3) \quad \exists t_0 > 0 \text{ s.t. } F(t_0) > 0.$$

3.1.2. Critical case.

- Assumptions on $a(x)$. Here we assume $b(x) \equiv 0$, that is, $a(x) \equiv V(x)$.

$$(V_1^*) \quad \begin{cases} V(x) \in L_{\text{loc}}^1(\mathbb{R}^N) \cap C(\mathbb{R}^N \setminus \mathcal{O}), \text{ where } \mathcal{O} \text{ is a finite set. } V(x) \leq 0 \text{ a.e. } x \in \mathbb{R}^N. \\ C_V^* := \inf_{u \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + V(x) u^2 \, dx}{\int_{\mathbb{R}^N} |V(x)| u^2 \, dx} > 0, \end{cases}$$

$$(V_2^*) \quad \begin{cases} \exists a_* \in \mathbb{R}^N \text{ s.t. } \begin{cases} V_0 = \lim_{|x| \rightarrow \infty} V(x) = 0, \\ V_+(x) = \lim_{\lambda \rightarrow \infty} \lambda^{-2s} V(\lambda^{-1}(x + a_*)), \text{ uniformly in compact sets} \\ V_-(x) = \lim_{\lambda \rightarrow 0} \lambda^{-2s} V(\lambda^{-1}(x + a_*)). \end{cases} \\ V_{\pm}(x) \text{ satisfies } (V_1^*), \text{ provided } V_{\pm}(x) \not\equiv 0. \end{cases}$$

$$(V_3^*) \quad \begin{cases} \forall (\lambda_k) \subset \mathbb{R}^+ \text{ s.t. either } |\lambda_k| \rightarrow \infty \text{ or } |\lambda_k| \rightarrow 0; \text{ and } (y_k) \subset \mathbb{R}^N \text{ s.t. } |\lambda_k y_k| \rightarrow \infty, \\ \lim_{k \rightarrow \infty} \lambda_k^{-2s} V(\lambda_k^{-1} x + y_k) = 0, \text{ uniformly in compact sets.} \end{cases}$$

- Assumptions on $f(x, t)$.

$$(f_1^*) \quad \begin{cases} f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R} \text{ is a Carathéodory function satisfying the growth condition,} \\ \exists C > 0, \text{ s.t. } |f(x, t)| \leq C|t|^{2s-1} \text{ a.e. } x \in \mathbb{R}^N, \forall t \in \mathbb{R}. \end{cases}$$

$$\begin{aligned}
(f_2^*) \quad & \left\{ \begin{array}{l} \forall a_1, \dots, a_M \in \mathbb{R}, \exists C = C(M) > 0 \text{ s.t.} \\ \left| F\left(x, \sum_{n=1}^M a_n\right) - \sum_{n=1}^M F(x, a_n) \right| \leq C(M) \sum_{m \neq n \in \{1, \dots, M\}} |a_n|^{2_s^*-1} |a_m| \quad \text{a.e. } x \in \mathbb{R}^N. \end{array} \right. \\
(f_3^*) \quad & \left\{ \begin{array}{l} \exists \gamma > 1 \text{ s.t. the limits are uniform convergent in } x \text{ and in compact sets for } t : \\ f_0(t) := \lim_{|x| \rightarrow \infty} f(x, t), \\ f_+(t) := \lim_{j \in \mathbb{Z}, j \rightarrow +\infty} \gamma^{-\frac{N+2s}{2}j} f\left(\gamma^{-j}x, \gamma^{\frac{N-2s}{2}j}t\right), \\ f_-(t) := \lim_{j \in \mathbb{Z}, j \rightarrow -\infty} \gamma^{-\frac{N+2s}{2}j} f\left(\gamma^{-j}x, \gamma^{\frac{N-2s}{2}j}t\right), \\ \text{and the primitive } F_\kappa(t) \text{ satisfies } (f_3') \text{ for each } \kappa = 0, +, -. \end{array} \right.
\end{aligned}$$

$$(f_4^*) \quad \forall \kappa = 0, +, -, \text{ the function } t \mapsto \frac{f_\kappa(t)}{|t|}, \text{ is strict increasing in } \mathbb{R}.$$

From (V_1^*) we can see that $\|\cdot\|_V$ defines a norm in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ which is equivalent to the standard one (see Proposition 6.1). Thus in the critical case we consider associated with problem (\mathcal{P}_s) the energy functional $I_* : \mathcal{D}^{s,2}(\mathbb{R}^N) \rightarrow \mathbb{R}$ given by

$$I_*(u) = \frac{1}{2} \|u\|_V^2 - \int_{\mathbb{R}^N} F(x, u) dx, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

which is well defined and is continuously differentiable provided that (f_1^*) holds. We can define $c(I_*)$ and Γ_{I_*} similarly as in (3.2) and (3.3), by just replacing $H_V^s(\mathbb{R}^N)$ by $\mathcal{D}^{s,2}(\mathbb{R}^N)$. We consider the next assumption in order to compare the minimax level of the associated energy functional of Eq. (\mathcal{P}_s) with the one resulting limiting energy functionals.

$$(\mathcal{H}^*) \quad \left\{ \begin{array}{l} V(x) \leq V_\pm(x), \text{ a.e. } x \in \mathbb{R}^N, \\ \forall \kappa = 0, +, -, F_\kappa(t) \leq F(x, t), \text{ a.e. } x \in \mathbb{R}^N, \forall t \in (-\delta, \delta). \end{array} \right.$$

$$(\mathcal{H}_0^*) \quad \left\{ \begin{array}{l} \text{The first inequality in } (\mathcal{H}^*) \text{ is strict for a set of positive measure or} \\ \exists \delta > 0 \text{ s.t. the second inequality in } (\mathcal{H}^*) \text{ is strict a.e. } x \in \mathbb{R}^N, \forall t \in (-\delta, \delta). \end{array} \right.$$

We also consider the autonomous case, that is, $f(x, t) \equiv f(t)$. To study this case we assume that the nonlinearity is self-similar in the sense of [22], more precisely,

$$(f_5^*) \quad \exists \gamma > 1 \text{ s.t. } F(t) = \gamma^{-Nj} F\left(\gamma^{\frac{N-2s}{2}j} t\right), \quad \forall t \in \mathbb{R}, j \in \mathbb{Z}.$$

3.2. Statement of the main existence results. We first state our results concerning existence of ground state solutions for Eq. (\mathcal{P}_s) in both subcritical and critical growth range of the nonlinearity.

Theorem 3.1.

- (i) Suppose that $f(x, t)$ and $a(x) \equiv V(x)$ are 1-periodic in each $x_i, i = 1, \dots, N$ and satisfies (f_1) – (f_3) or (f_3) – (f_6) and (V_1) – (V_2) respectively. Then the equation (\mathcal{P}_s) has a ground state solution.

(ii) Suppose that $f(t) \in C^1(\mathbb{R}^N)$ satisfies (f'_3) and (f_5^*) for some $\gamma > 1$. Let

$$\mathcal{G} = \left\{ u \in \mathcal{D}^{s,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} F(u) \, dx = 1 \right\},$$

and consider

$$\mathcal{I}_\lambda = \inf_{u \in \mathcal{G}} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 - \lambda |x|^{-2s} u^2 \, dx, \quad (3.4)$$

where $0 < \lambda < \Lambda_{N,s}$ is given by (1.3). Then, there is a radial minimizer w for (3.4). Furthermore, there exists $\alpha > 0$ such that $u = w(\cdot/\alpha)$ is a ground state solution for (\mathcal{P}_s) , with $a(x) = -\lambda|x|^{-2s}$.

Theorem 3.1 take account the invariance of I under the action of translations and dilations in $H^s(\mathbb{R}^N)$ and $\mathcal{D}^{s,2}(\mathbb{R}^N)$, to obtain the concentration-compactness of Palais-Smale and minimizing sequences in each case respectively. These properties are sufficient to ensure existence of ground state solutions of (\mathcal{P}_s) . Moreover, our results improve and complement [15] for the fractional framework since here we consider potential $a(x)$ and nonlinearity $F(x, t)$ which can change sign. Also in Theorem 3.1 (ii) we do not require the classical Ambrosetti-Rabinowitz condition (f_2) . Our argument to prove Theorem 3.1 (ii) involves a Pohozev type identity and as usual, for this we require C^1 regularity of $f(t)$.

Theorem 3.2. *Let*

$$\bar{c}(I) := \inf_{u \in H_V^s(\mathbb{R}^N) \setminus \{0\}} \sup_{t \geq 0} I(tu) \quad \text{and} \quad c_{\mathcal{N}}(I) := \inf_{u \in \mathcal{N}} I(u),$$

where $\mathcal{N} = \{u \in H_V^s(\mathbb{R}^N) \setminus \{0\} : I(u) \cdot u = 0\}$. Suppose that for a.e. $x \in \mathbb{R}^N$ the function

$$t \mapsto \frac{f(x, t)}{|t|} \quad \text{is strict increasing in } \mathbb{R}. \quad (3.5)$$

If $V(x) \in L^1_{\text{loc}}(\mathbb{R}^N)$, $a(x) = V(x) - b(x)$ satisfies (V_2) – (V_3) and $f(x, t)$ fulfills (f_1) – (f_2) , then

$$c(I) = \bar{c}(I) = c_{\mathcal{N}}(I).$$

In particular, any nontrivial weak solution u in $H_V^s(\mathbb{R}^N)$ at the mountain pass level is a ground state solution.

Notice that in Theorem 3.2 we deal with the case where $a(x)$ changes sign and is not necessarily bounded from below, also with nonlinearity having the behavior at 0 described by (f'_1) . Moreover, as we can see, Theorem 3.2 proves existence of ground state solution by replacing the aforementioned invariance by the monotonicity (3.5). In fact, our results below give some conditions that guarantee existence of nontrivial weak solutions in $H_V^s(\mathbb{R}^N)$ at the mountain pass level. Therefore, Theorem 3.2 improves some results in [52].

Our next results are concerned with existence of nontrivial weak solutions for (\mathcal{P}_s) by means of concentration-compactness of Palais-smale sequences at the mountain pass level.

Theorem 3.3. *Assume that (f_1) – (f_3) or (f_3) – (f_6) hold; and additionally (f_7) . Suppose also that $a(x)$ and $f(x, t)$ satisfies either one of the following conditions*

- (i) $b(x) \equiv 0$, (V_1) – (V_2) , (f_8) and (f_{10}) ; or
- (ii) $V(x) \geq 0$, $b(x)$ has compact support, (V_2) – (V_4) , (f_9) and (f'_{10}) ; or
- (iii) Replace conditions (f_{10}) and (f'_{10}) in the above items by

$$I(u) \leq I_{\mathcal{P}}(u) \quad \text{and} \quad I(u) \leq I_\infty(u), \quad \forall u \in H_V^s(\mathbb{R}^N), \quad (3.6)$$

respectively for each considered case.

Then Eq. (\mathcal{P}_s) possess a nontrivial weak solution u in $H_V^s(\mathbb{R}^N)$ at the mountain pass level, that is, $I(u) = c(I)$. Moreover, under the assumptions of items (i) and (ii), any sequence (u_k) in $H_V^s(\mathbb{R}^N)$ such that $I(u_k) \rightarrow c(I)$ and $I'(u_k) \rightarrow 0$ has a convergent subsequence.

Theorems 3.1 (i) and 3.3 extend and complement the existence results of [15, 52, 64] in the fractional framework. Observe also that in Theorem (ii) the potential $a(x) = V(x) - b(x)$ is not necessarily bounded from below and we do not ask condition (f_8) as it was made in the previously mentioned references.

Theorem 3.4. Assume that $f(x, t)$ and $a(x) \equiv V(x)$ satisfies (f_1^*) – (f_4^*) , (\mathcal{H}^*) , (f_2) – (f_3) and (V_1^*) – (V_3^*) respectively. Then Eq. (\mathcal{P}_s) has a nontrivial weak solution in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ at the mountain pass level. If we assume additionally condition (\mathcal{H}_0^*) , then any sequence (u_k) in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ such that $I_*(u_k) \rightarrow c(I_*)$ and $I'_*(u_k) \rightarrow 0$ has a convergent subsequence.

Theorems 3.1 (ii) and 3.4 complement the study made in [13, 20]. Theorem 3.4 can be seen as an nonlocal version of [13, Theorem 5.2], since we take account that the critical nonlinearity is not autonomous. It also can be seen as a complement for many results in the literature about existence of nontrivial weak solution for Schrödinger equation involving critical nonlinearities and singular potentials (cf. [26, 27, 57, 61] and the references given there).

3.2.1. Remark on the hypothesis.

Remark 3.5. Some comments on our assumptions are in order.

- (i) Assumption (f_1) can be seen as a subcritical version of (f_5^*) in the sense that it is oscillating about a subcritical power $|t|^{p-2}t$, $2 < p < 2_s^*$. In fact, it is easy to see that (f_1) holds if $f(x, t)$ satisfies one of the following conditions:

$$(f_1') \quad \lim_{t \rightarrow 0} \frac{f(x, t)}{|t| + |t|^{2_s^*-1}} = 0, \quad \text{uniform in } x.$$

$$(f_1'') \quad \begin{cases} \exists \varrho(t) \in C(\mathbb{R} \setminus \{0\}) \cap L^\infty(\mathbb{R}) \text{ s.t. } 2 < \inf_{t \in \mathbb{R}} \varrho(t), \sup_{t \in \mathbb{R}} \varrho(t) < 2_s^* \text{ and} \\ |f(x, t)| \leq C(1 + |t|^{\varrho(t)-1}) \quad \text{a.e. } x \in \mathbb{R}^N, \forall t \in \mathbb{R}; \end{cases}$$

For example, $f(x, t) = k(x) [\varrho'(t)(\ln |t|) + \varrho(t)] |t|^{\varrho(t)-2}t$, $f(x, 0) \equiv 0$, satisfies (f_1') and (f_1'') , where

$$\varrho(t) = \frac{2_s^* - 2}{16} \sin(\ln(|\ln |t||)) + \frac{52_s^* + 6}{8} \quad \text{and} \quad 0 \leq k(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}^N).$$

A version of (f_1) for the local case appeared in [62].

- (ii) It follows by the same arguments found in [15, Lemma 2.1] that (f_4) and (f_6) imply that there exists $p \in (2, 2_s^*)$ such that

$$\forall \varepsilon, \exists C_\varepsilon > 0 \text{ s.t. } |f(x, t)| \leq \varepsilon |t| + C_\varepsilon |t|^{p-1}, \quad \text{a.e. } x \in \mathbb{R}^N, \forall t \in \mathbb{R}.$$

Note that this is a special case of (f_1) , precisely when $p_\varepsilon = p$.

- (iii) Assumption (f_2) is the well-known Ambrosetti-Rabinowitz condition which is used to obtain the mountain pass geometry and the boundedness of Palais-Smale sequences for the associated functional (see for instance [1, 47]). Conditions (f_4) – (f_6) are an alternative for f_2 , and was first introduced in [19] for the local case. By similar arguments in [19],

condition (f_6) holds if we assume (f_4) , (f_5) and that there exists $p \in (2, 2_s^*)$ and $c_1, c_2, r_1 > 0$ such that

$$|f(x, t)| \leq c_1 |t|^{p-1} \quad \text{and} \quad F(x, t) \leq \left(\frac{1}{2} - \frac{1}{c_2 |t|^\nu} \right) f(x, t)t, \quad \forall |t| \geq r_1.$$

where $1 < \nu < 2$ if $N = 1$, and $1 < \nu < N + p - pN/2s$ if $N \geq 2$.

- (iv) In view of the boundedness of Palais-Smale sequences we point out that we separate our studies for the subcritical case in two distinct situations: $f(x, t)$ satisfies (f_1) – (f_3) or (f_3) – (f_6) . The first one is associated to the case where $f(x, t)$ has an oscillatory behavior around the subcritical power and the second one refers to the case where $f(x, t)$ does not satisfies the Ambrosetti-Rabinowitz condition.
- (v) In [15], considering the local case of Schrödinger equations with asymptotically periodic terms, is was proved the mountain pass geometry assuming $F(x, t) > 0$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ and (f_4) instead of the classical Ambrosetti-Rabinowitz condition. Here, in this work, we have an improvement even to the local case because we assume (f_3) instead of assuming that $F(x, t) > 0$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.
- (vi) Assumption (f_5) it is used to prove the boundedness of Palais-Smale sequences at the mountain pass level for the functional of Eq. (\mathcal{P}_s) . In [15] to prove similar result the author assumed the following more restrictive condition $\mathcal{F}(x, t) = \frac{1}{2}f(x, t)t - F(x, t) \geq b(t)t^2$, for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, for some $b(t) \in C(\mathbb{R} \setminus \{0\}, \mathbb{R}^+)$.
- (vii) In our approach to study existence of weak solutions of Eq. (\mathcal{P}_s) we use assumption (f_7) , unlike the aforementioned papers, where the authors impose the more tight condition $|f(x, t) - f_{\mathcal{P}}(x, t)| \leq h(x)|t|^{q-1}$, a.e. x in \mathbb{R}^N and for all t in \mathbb{R} , where $h(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\{x \in \mathbb{R}^N : |h(x)| \geq \varepsilon\}$ has finite Lebesgue measure, for any $\varepsilon > 0$.
- (viii) The smoothness condition assumed in (f_9) is the natural hypothesis used in the literature to prove that weak solutions of Eq. (\mathcal{P}_s) satisfies a Pohozaev type identity.
- (ix) We prove in Proposition 6.1 that $H_V^s(\mathbb{R}^N)$ is well defined and it is continuously embedded in $H^s(\mathbb{R}^N)$. As a consequence of this we can conclude that the infimum $\mathcal{C}_V^{(\beta)}$ defined in (V_3) is strictly positive.
- (x) The class of functions satisfying (f_5^*) can be seen as nonlinearities asymptotically oscillating about the critical power $|t|^{2_s^*-2}t$ and was introduced in [22, 63].
- (xi) The asymptotic additivity given in (f_2^*) ensure the convergence of functional I under the weak profile decomposition for bounded sequences in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ described in Theorem A (for more details see also [22]).
- (xii) As already mentioned, conditions (V_1^*) – (V_3^*) define a class of singular potentials that vanishes at infinite. See Example 3.7–(iv).
- (xiii) Once the limits in (V_4) , (f_7) , (f_9) or (f_3^*) exist, to obtain compactness of Palais-Smale sequences at the minimax levels we need to require the additional conditions over the minimax levels $c_{\mathcal{P}}, c_\infty, c_0, c_+, c_-$ given in assumptions (f_{10}) , (f'_{10}) , (\mathcal{H}^*) – (\mathcal{H}_0^*) . In fact, we do not believe that it is possible, in general, to achieve the compactness described in Theorems 3.3 and 3.4 without these conditions. We mention that this kind of approach was introduced by P.-L. Lions in [39–42].
- (xiv) We also consider the case when (f_{10}) , (f'_{10}) , (\mathcal{H}^*) – (\mathcal{H}_0^*) do not hold. Precisely, when it is allowed $c(I) = c(I_{\mathcal{P}})$ or $c(I) = c(I_\infty)$. In this case, the concentration-compactness argument at the mountain pass level cannot be used. We apply [38, Theorem 2.3] to overcome this difficulty and prove existence of solution at the mountain pass level.

- (xv) For problem (\mathcal{P}_s) involving critical growth we require conditions (V_1^*) – (V_3^*) on the potential and (f_3^*) – (\mathcal{H}_0^*) on the nonlinear term $f(x, t)$. These assumptions are suitable for our argument, differently from (f_{10}) – (f'_{10}) , because the potential that appears in the associated limiting equation depends on the profile decomposition of Theorem A for a given Palais-Smale sequence at the mountain pass level (for more details see estimate (10.1)).

Remark 3.6. Under the assumptions (V_4) and (f_7) we describe the next conditions which imply that (f_{10}) and (f'_{10}) hold:

$$(\mathcal{H}) \begin{cases} F_{\mathcal{P}}(x, t) \leq F(x, t), & \text{a.e. } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}, \\ V(x) \leq V_{\infty}, & \text{a.e. } x \in \mathbb{R}^N. \\ \text{The first inequality holds strictly in some open interval contained the origin} \\ \text{or the second one holds in a set of positive measure.} \end{cases}$$

In Proposition 9.1, under suitable conditions, we obtained the following estimates for the minimax levels, $c(I) \leq c(I_{\mathcal{P}})$ and $c(I) \leq c(I_{\infty})$. Moreover, we proved that under condition (\mathcal{H}) we have that (f_{10}) and (f'_{10}) hold. We observe that on the corresponding assumption of Theorem 3.3, it is easy to see that the inequalities in (\mathcal{H}) imply that (3.6) is satisfied.

Example 3.7. Our approach include the following classes of potentials:

- (i) Let $w \in C(\mathbb{R}^N)$ with $w(x) > w_0 > 0$ for all $x \in \mathbb{R}^N$. For a potential $a(x) \equiv V(x)$ satisfying (V_1) – (V_2) define $V(x) \in C(\mathbb{R}^N)$ 1-periodic in each x_i , $i = 1, \dots, N$, such that

$$V(x) = \begin{cases} w(x), & \text{if } x \in Q_1 \setminus B_{1/4}, \\ 0, & \text{if } x \in B_{1/4}, \end{cases}$$

where $Q_1 = \{x \in \mathbb{R}^N : -1 \leq x_i \leq 1, i = 1, \dots, N\}$ is a N -dimensional cube.

- (ii) For a potential $a(x) \equiv V(x)$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ fulling condition (V_2) we may take any $V(x) \in C(\mathbb{R})$ such that $V(x)$ is nonpositive in a ball B_R with radius R in \mathbb{R}^N and is nonnegative outside of it.
- (iii) To study potential of the form $a(x) = V(x) - b(x)$, setting

$$V(x) = 2 - \frac{1}{1 + |x|^2} \quad \text{and} \quad V_{\infty} = 2,$$

and

$$b(x) = \begin{cases} \mathcal{C}_b |x|^{-\delta}, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1, \end{cases}$$

we can verify that $a(x) = V(x) - b(x)$ satisfies conditions (V_2) – (V_4) . Here \mathcal{C}_b is a positive normalization constant, $0 < \delta < N/\beta$ and $\beta > N/2s$.

- (iv) For potential $a(x) \equiv V(x)$ satisfying assumptions (V_1^*) – (V_3^*) we can consider

$$V(x) = -\frac{1}{L} \sum_{j=1}^L \frac{\lambda_j}{|x - x^j|^{2s}}, \quad \text{with } 0 < \lambda_j < \frac{\Gamma_{N,s}}{2}, \quad j = 1, \dots, L,$$

which is well defined in view of (1.3).

Example 3.8. Note that the hypotheses of Theorems 3.1–3.4 are for example satisfied by nonlinearities of the following forms:

- (i) Let $\varrho(t)$ be as in Remark 3.5–(i) and consider $k(x) = |x|^2/(1 + |x|^2)$. One can see that

$$f(x, t) = k(x) [\varrho'(t)(\ln |t|) + \varrho(t)] |t|^{\varrho(t)-2}t, \quad f(x, 0) \equiv 0,$$

satisfies assumptions (f_1) – (f_3) , (f_9) and (f'_{10}) .

- (ii) For a nonlinearity satisfying conditions (f_3) – (f_6) , (f_7) , (f_8) and (f_{10}) we can define

$$f(x, t) = \begin{cases} h(x, t), & \text{for } t \geq 0, \\ -h(x, -t), & \text{for } t < 0, \end{cases}$$

where

$$h(x, t) = k(x)t \ln(1 + t) + k_1(x) [(1 + \cos(t))t^2 + 2(t + \sin(t))t],$$

for $t \geq 0$, $s > N/6$; $k(x) = |x|^2/(1 + |x|^2)$ and $0 \leq k_1(x) \in C(\mathbb{R}^N)$ is such that $\lim_{|x| \rightarrow \infty} k_1(x) = 0$.

- (iii) Let $c(x)$ be a continuous nonnegative 1–periodic function in each x_i , $i = 1, \dots, N$, and consider $f(x, t) = c(x) [ph_\varepsilon(t) + h'_\varepsilon(t)t] |t|^{p-1}$, $2 < p < 2_s^*$, and $h_\varepsilon(t) \in C^\infty(\mathbb{R})$ is a nondecreasing cutoff function satisfying

$$\begin{cases} |h'_\varepsilon(t)| \leq C/t, \quad |h_\varepsilon(t)| \leq C, \quad \forall t \in \mathbb{R}, \\ h_\varepsilon(t) = -\varepsilon, \text{ for } t \leq 1/4, \quad h_\varepsilon(t) = \varepsilon, \text{ for } t \geq 1/4, \text{ with } \varepsilon \text{ small enough.} \end{cases}$$

We empathize the fact that $F(x, t)$ changes sign.

- (iv) Suppose that the function $k_0(x)$ is continuous and

$$2_s^* - \mu > \sup_{x \in \mathbb{R}^N} k_0(x) \geq k_0(x) > k_0(0) = \inf_{x \in \mathbb{R}^N} k_0(x) = \lim_{|x| \rightarrow \infty} k_0(x) = 0.$$

The nonlinearity given below satisfies the hypothesis of Theorem 3.4,

$$f(x, t) = \exp\{k_0(x)(\sin(\ln |t|) + 2)\} [k_0(x) \cos(\ln |t|) + 2_s^*] |t|^{2_s^*-2}t, \quad f(x, 0) \equiv 0.$$

4. PRELIMINARIES

4.1. Fractional Sobolev spaces. Let $0 < s < N/2$, by Placherel Theorem, we have

$$[u]_s^2 = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N).$$

Thus, the space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is well defined with continuous embedding

$$\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N) \quad \text{for } 0 < s < N/2,$$

in view of the well know inequality

$$\int_{\mathbb{R}^N} |u|^{2_s^*} dx \leq \mathcal{K}_* \left(\int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u|^2 d\xi \right)^{2_s^*/2}, \quad \forall u \in C_0^\infty(\mathbb{R}^N),$$

where

$$\mathcal{K}_* = \left[2^{-2s} \frac{\Gamma(\frac{N-2s}{2})}{\Gamma(\frac{N+2s}{2})} \left(\frac{\Gamma(N)}{\Gamma(N/2)} \right)^{2s/N} \right]^{2_s^*/2}$$

and $\mathcal{F}u$ is defined in (1.1). Moreover, $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as a separable Hilbert space when endowed with the inner product

$$[u, v]_s = \int_{\mathbb{R}^N} (-\Delta)^{s/2}u (-\Delta)^{s/2}v dx, \quad \forall u, v \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

as well the following characterization hold

$$\begin{aligned}\mathcal{D}^{s,2}(\mathbb{R}^N) &= \left\{ u \in L^{2^*}(\mathbb{R}^N) : (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N) \right\} \\ &= \left\{ u \in L^{2^*}(\mathbb{R}^N) : |\xi|^s \mathcal{F}u \in L^2(\mathbb{R}^N) \right\}.\end{aligned}$$

Let $\Omega \subset \mathbb{R}^N$ be an open set and $0 < s < 1$, the inhomogeneous fractional Sobolev space is defined as

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\}, \quad (4.1)$$

with the norm

$$\|u\|_{H^s(\Omega)}^2 := \int_{\Omega} u^2 dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy.$$

By [18, Proposition 3.4], we have

$$[u]_s^2 = \frac{C(N, s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy, \quad \text{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

where the positive constant $C(N, s)$ is given in (1.2). Thus, since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^s(\Omega)$ when $\Omega = \mathbb{R}^N$, we have

$$\begin{aligned}H^s(\mathbb{R}^N) &= \{ u \in L^2(\mathbb{R}^N) : |\xi|^s \mathcal{F}u \in L^2(\mathbb{R}^N) \} \\ &= \left\{ u \in L^2(\mathbb{R}^N) : (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N) \right\},\end{aligned} \quad (4.2)$$

for $0 < s < 1$. Turns out that definition of $H^s(\mathbb{R}^N)$ given in (4.2) it is more appropriated for the general case $s \geq 0$, than definition (4.1), because for $s \geq 1$, the integral in (4.1) is finite if and only if u is constant (see [5, Proposition 2]). Consequently, we can consider the following inner product and the associated norm in $H^s(\mathbb{R}^N)$,

$$(u, v) := \int_{\mathbb{R}^N} (-\Delta)^{s/2}u (-\Delta)^{s/2}v + uv dx, \quad \|u\|^2 := \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx + u^2 dx.$$

Moreover, for Ω an open set of class $C^{0,1}$ with bounded boundary, we have the continuous embedding

$$H^s(\Omega) \hookrightarrow L^p(\Omega), \quad 2 \leq p \leq 2_s^*, \quad \text{for } 0 < s < N/2, \quad (4.3)$$

and the following compact embedding (see [18, Section 7]),

$$\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L_{\text{loc}}^p(\mathbb{R}^N), \quad 1 \leq p < 2_s^*, \quad \text{for } 0 < s < \min\{1, N/2\}. \quad (4.4)$$

Thus, every bounded sequence in $H^s(\mathbb{R}^N)$ has subsequence that converges strongly in $L^p(\Omega)$, $1 \leq p < 2_s^*$, for any compact set Ω of \mathbb{R}^N .

We finish this section emphasizing that the Plancherel Theorem also gives the next identity, which is used several times throughout this paper

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2}u (-\Delta)^{s/2}v dx = \int_{\mathbb{R}^N} (-\Delta)^s uv dx, \quad \forall u \in H^{2s}(\mathbb{R}^N), v \in H^s(\mathbb{R}^N). \quad (4.5)$$

4.2. The s -harmonic extension. Next for the reader convenience we introduce the harmonic extension, following [33, Section 2] and for that we begin describing a class of weight Sobolev spaces suitable to work with this harmonic extension. First, observe that, for any $0 < s < 1$, the function $z = (x, y) \mapsto |y|^{1-2s}$ belongs to the Muckenhoupt class \mathcal{A}_2 of weights in \mathbb{R}^{N+1} , that is

$$\left(\frac{1}{|B|} \int_B |y|^{1-2s} dx dy \right) \left(\frac{1}{|B|} \int_B |y|^{2s-1} dx dy \right) \leq C, \quad \text{for all ball } B \text{ in } \mathbb{R}^{N+1}.$$

More details can be found in [23]. Let Q be a open set in \mathbb{R}^{N+1} , we consider $L^2(Q, |y|^{1-2s})$ as the Banach space of the Lebesgue measurable functions v defined in Q such that

$$\|v\|_{L^2(Q, |y|^{1-2s})} = \left(\int_Q |y|^{1-2s} v^2 dx dy \right)^{1/2} < \infty.$$

We also consider the space $H^1(Q, |y|^{1-2s})$ of the functions w in $L^2(Q, |y|^{1-2s})$ such that the weak derivatives w_{z_i} exist and belong to $L^2(Q, |y|^{1-2s})$ for $i = 1, \dots, N+1$. It is easy to see that $H^1(Q, |y|^{1-2s})$ is a Hilbert space when endowed with the inner product

$$(v_1, v_2)_{H^1(Q, |y|^{1-2s})} = \int_Q |y|^{1-2s} \langle \nabla v_1, \nabla v_2 \rangle + |y|^{1-2s} v_1 v_2 dx dy,$$

and the induced norm

$$\|v\|_{H^1(Q, |y|^{1-2s})} = \left(\int_Q |y|^{1-2s} |\nabla v|^2 + |y|^{1-2s} v^2 dx dy \right)^{1/2}.$$

We call attention to the fact that the space of smooth functions $C^\infty(Q) \cap H^1(Q, |y|^{1-2s})$ is dense in the weight Sobolev space $H^1(Q, |y|^{1-2s})$ (see [66] for further details).

Regarding the space $H^1(Q, y^{1-2s})$ with $Q = \Omega \times (0, R)$, where $\Omega \subset \mathbb{R}^N$ is a domain with Lipschitz boundary, it is well know the existence of a well-defined trace operator

$$t_r : H^1(Q, y^{1-2s}) \rightarrow H^s(\Omega)$$

with

$$\|t_r(v)\|_{H^s(\Omega)} \leq C \|v\|_{H^1(Q, y^{1-2s})}, \quad \forall v \in H^1(Q, y^{1-2s}),$$

where $C > 0$, depends only on N, s and Ω (see also [44]). Moreover, by the continuous embedding $H^s(\Omega) \hookrightarrow L^{2^*_s}(\Omega)$, we have

$$\|t_r(v)\|_{L^{2^*_s}(\Omega)} \leq C \|v\|_{H^1(Q, y^{1-2s})}, \quad \forall v \in H^1(Q, y^{1-2s}). \quad (4.6)$$

Let

$$P_s(x, y) = \beta(N, s) \frac{y^{2s}}{(|x|^2 + y^2)^{\frac{N+2s}{2}}},$$

where $\beta(N, s)$ is such that $\int_{\mathbb{R}^N} P_s(x, 1) dx = 1$ and $0 < s < 1$. Considering the standard notation

$$\mathbb{R}_+^{N+1} = \{(x, y) \in \mathbb{R}^{N+1} : y > 0\},$$

for $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ let us set the s -harmonic extension of u ,

$$w(x, y) = E_s(u)(x, y) := \int_{\mathbb{R}^N} P_s(x - \xi, y) u(\xi) d\xi, \quad (x, y) \in \mathbb{R}_+^{N+1}.$$

Then, for any compact subset K of $\overline{\mathbb{R}_+^{N+1}}$, we have $w \in L^2(K, y^{1-2s})$, $\nabla w \in L^2(\mathbb{R}_+^{N+1}, y^{1-2s})$ and $w \in C^\infty(\mathbb{R}_+^{N+1})$. Moreover, w satisfies

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla w) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ -\lim_{y \rightarrow 0^+} y^{1-2s}w_y(x, y) = \kappa_s(-\Delta)^s u(x) & \text{in } \mathbb{R}^N, \\ \|\nabla w\|_{L^2(\mathbb{R}_+^{N+1}, y^{1-2s})}^2 = \kappa_s \|u\|^2, \end{cases} \quad (4.7)$$

where we understand (4.7) in the distribution sense, where $\kappa_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$, and Γ is the gamma function. Precisely,

$$\int_{B_R^+} y^{1-2s} \langle \nabla w, \nabla \varphi \rangle \, dx dy = \kappa_s \int_{B_R^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} (t_r \varphi) \, dx, \quad \forall \varphi \in C_0^\infty(B_R^+ \cup B_R^N),$$

where for $R > 0$,

$$\begin{cases} B_R = \{z = (x, y) \in \mathbb{R}^{N+1} : |z|^2 < R^2\}, \\ B_R^+ = B_R \cap \mathbb{R}_+^{N+1} \text{ and} \\ B_R^N = \{z = (x, y) \in \mathbb{R}_+^{N+1} : |z|^2 < R^2, y = 0\}. \end{cases} \quad (4.8)$$

More generally, given $g : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ we say that a function $v \in H^1(B_R^+, y^{1-2s})$ is a weak solution of the problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } B_R^+, \\ -\lim_{y \rightarrow 0^+} y^{1-2s}v_y(x, y) = \kappa_s g(t_r(v)(x)) & \text{in } B_R^N, \end{cases} \quad (4.9)$$

if, for all $\varphi \in C_0^\infty(B_R^+ \cup B_R^N)$, we have

$$\int_{B_R^+} y^{1-2s} \langle \nabla v, \nabla \varphi \rangle \, dx dy = \kappa_s \int_{B_R^N} g(t_r(v)) t_r(\varphi) \, dx. \quad (4.10)$$

Suppose that $g(x, t) = f(x, t) - a(x)t$ and let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be such that $f(u)$ and $F(u)$ belongs to $L^1(\mathbb{R}^N)$. Let us assume also that $V(x)$ belong to $L_{\text{loc}}^1(\mathbb{R}^N)$, satisfies (V₂) and that $b(x)$ verifies (V₃). Then $w = E_s(u)$ is a weak solution of (4.9) for all R if, and only if, u is a weak solution of (P_s).

Remark 4.1. Using the s -harmonic extension, it can be proved the existence of nonnegative weak solutions of (P_s) if $f(x, t) \geq 0$ for all $t \geq 0$ and a.e. x in \mathbb{R}^N . For that one can consider the truncation

$$\bar{f}(x, t) = \begin{cases} f(x, t), & \text{if } t \geq 0 \\ 0, & \text{if } t < 0. \end{cases}$$

Assume that $a(x) \in L_{\text{loc}}^1(\mathbb{R}^N)$ and that conditions (f₁) and (V₂) hold true with $b(x) \equiv 0$. Thus for u a weak solution of (P_s), with $f(x, t)$ replaced by $\bar{f}(x, t)$, we have that u is also a weak nonnegative solution for (P_s). To see that, let $\xi \in C_0^\infty(\mathbb{R} : [0, 1])$ such that

$$\xi(t) = \begin{cases} 1, & \text{if } t \in [-1, 1] \\ 0, & \text{if } |t| \geq 2 \end{cases} \quad \text{and} \quad |\xi'(t)| \leq C, \quad \forall t \in \mathbb{R},$$

for some C positive constant. For each $n \in \mathbb{N}$, define $\xi_n : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ by $\xi_n(z) = \xi(|z|^2/n^2)$. Then $\xi_n \in C_0^\infty(\mathbb{R}^{N+1})$ and verifies

$$|\nabla \xi_n(z)| \leq C \quad \text{and} \quad |z| |\nabla \xi_n(z)| \leq C, \quad \forall z \in \mathbb{R}^{N+1}.$$

By a density argument, we can take $\varphi = \xi_n w_-$ in (4.10), where $w_-(z) = \min\{w(z), 0\}$. Since $w_-(z) = E_s(u_-)$, we have that

$$\begin{aligned} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \xi_n |\nabla w_-|^2 + y^{1-2s} \xi_n \langle \nabla w_+, \nabla w_- \rangle + y^{1-2s} \langle \nabla w_+ + \nabla w_-, w_- \nabla \xi_n \rangle \, dx dy \\ = \kappa_s \int_{\mathbb{R}^N} (\bar{f}(x, u) - a(x)u) \xi_n u_- \, dx, \end{aligned}$$

and we may apply the Dominated Convergence Theorem and (4.7) to get

$$\|u_-\|_V^2 = \int_{\mathbb{R}^N} \bar{f}(x, u) u_- \, dx = 0.$$

Which implies that $u_- = 0$. On the other hand, if u has sufficient regularity one can show u is positive, by applying the maximum principle for the fractional Laplacian as described in [55]. In order to regularize the solutions of Eq. (\mathcal{P}_s), we follow the same arguments of [52, Section 6], but as already mentioned in this paper, we need sufficient regularity in the potential $a(x) \equiv V(x)$, which is beyond our scope.

4.3. D-weak convergence and dislocation spaces. As already mentioned, to achieve the decomposition described in Theorem 2.2, we follow the abstract approach of D -weak convergence and dislocation spaces developed in [65]. For convenience of the reader we include here some basic concepts and facts on this subject. In this subsection H denotes a Hilbert space.

Definition B. [65, Definition 3.1] Let D be a set of bounded linear operators on H , such that for every $g \in D$, $\inf_{u \in H, \|u\|=1} \|gu\| > 0$. We will say that a sequence $(u_k) \subset H$ converges to u in the D -weak sense in H , which we will denote as $u_k \xrightarrow{D} u$, in H , if for any sequence $(g_k) \subset D$, $(g_k^* g_k)^{-1} g_k^* (u_k - u) \rightarrow 0$ in H .

For (g_k) a sequence of bounded linear operators in H , we use the notation $g_k \rightarrow 0$ to indicate that $g_k u \rightarrow 0$ in H for all $u \in H$.

Definition C. [65, Definition 3.2] Assume that H is separable. A set D of bounded linear operators on H is a set of dislocations if

$$\begin{aligned} 0 < \delta := \inf_{g \in D, \|u\|=1} \|gu\|^2 \leq \sup_{g \in D, \|u\|=1} \|gu\|^2 < \infty, \\ (u_k) \subset H, (g_k) \subset D, u_k \rightarrow 0 \text{ in } H \Rightarrow g_k^* g_k u_k \rightarrow 0 \text{ in } H, \end{aligned}$$

and, whenever $(u_k) \subset H$ and $(g_k), (h_k) \subset D$,

$$h_k^* g_k \not\rightarrow 0, (g_k^* g_k)^{-1} g_k^* u_k \rightarrow 0 \text{ in } H \Rightarrow (h_k^* h_k)^{-1} h_k^* u_k \rightarrow 0 \text{ in } H.$$

The pair (H, D) is called a dislocation space.

The next result give a sufficient condition to establish if a pair (H, D) is a dislocation space.

Proposition D. [65, Proposition 3.1] Let H be a separable infinite-dimensional Hilbert space and D be a group (under the operator multiplication) of unitary operators $g : H \rightarrow H$, that is, $g^* = g^{-1}$. If

$$g_k \not\rightarrow 0 \text{ in } H, g_k \in D \Rightarrow g_k u \text{ has a convergent subsequence, } \forall u \in H,$$

then (H, D) is dislocation space.

The next result provides a profile decomposition for bounded sequence in a suitable abstract Hilbert space, it is crucial to obtain the decomposition in Theorem A, and it can be seen as a generalization of the celebrated Banach-Alaoglu-Bourbaki Theorem.

Theorem E. [65, Theorem 3.1] *Let (H, D) be a dislocation space. If $(u_k) \subset H$ is a bounded sequence, then there exists a set $\mathbb{N}_0 \subset \mathbb{N}$, and sequences $(w^{(n)})_{n \in \mathbb{N}_0} \subset H$, $(g_k^{(n)})_{k \in \mathbb{N}} \subset D$, $g_k^{(1)} = id$, with $n \in \mathbb{N}_0$, such that for a subsequence of (u_k) ,*

$$\begin{aligned} & \left(g_k^{(n)*} g_k^{(n)} \right)^{-1} g_k^{(n)*} u_k \rightharpoonup w^{(n)} \text{ in } H, \\ & g_k^{(n)*} g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m. \\ & \sum_{n \in \mathbb{N}_0} \|w^{(n)}\|^2 \leq \delta^{-1} \limsup_k \|u_k\|^2. \\ & u_k - \sum_{n \in \mathbb{N}_0} g_k^{(n)} w^{(n)} \xrightarrow{D} 0, \end{aligned}$$

where the series $\sum_{n \in \mathbb{N}_0} g_k^{(n)} w^{(n)}$ converges uniformly in k .

5. PROOF OF THEOREM 2.2

In this section we shall prove the mentioned profile decomposition for bounded sequences in $H^s(\mathbb{R}^N)$, $N \geq 2s$. From this, we derive further results concerning the compactness of the functional energy I . To achieve that we start by considering

$$D = D_{\mathbb{Z}^N} := \{g_y : H^s(\mathbb{R}^N) \rightarrow H^s(\mathbb{R}^N) : g_y u(x) = u(x - y), y \in \mathbb{Z}^N\},$$

which turns to be group of unitary operators in $H^s(\mathbb{R}^N)$. The idea is to obtain Theorem 2.2 by means of Theorem E. For that, we need first to determine how elements of $H^s(\mathbb{R}^N)$ becomes asymptotically orthogonal in $H^s(\mathbb{R}^N)$ with respect to any fixed other function under a sequence of dislocations.

Lemma 5.1. *Let be (y_k) a sequence in \mathbb{R}^N and $0 \neq u \in H^s(\mathbb{R}^N)$. The sequence $(u(\cdot - y_k))$ converges weakly to zero in $H^s(\mathbb{R}^N)$ if, and only if $|y_k| \rightarrow \infty$.*

Proof. Suppose first that $u(\cdot - y_k) \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$, and by contradiction, that $y_k \rightarrow y$ on a subsequence. By density we may assume that $u \in C_0^\infty(\mathbb{R}^N)$, also by [22, Lemma 5.1] we have that $u(\cdot - y_k) \rightarrow u(\cdot - y)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, consequently

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} (u(\cdot - y_k), u(\cdot - y)) \\ &= \lim_{k \rightarrow \infty} \left[\int_{\mathbb{R}^N} (-\Delta)^{s/2} u(\cdot - y_k) (-\Delta)^{s/2} u(\cdot - y) + u(\cdot - y_k) u(\cdot - y) dx \right] = \|u\|^2, \end{aligned} \quad (5.1)$$

where the convergence of the second term in (5.1) follows by the Dominated Convergence Theorem, which leads to a contradiction with the assumption that $u \neq 0$.

Conversely, assume that $|y_k| \rightarrow \infty$. Again, by density argument we may assume $u \in C_0^\infty(\mathbb{R}^N)$, and using [22, Lemma 5.2] we obtain that $u(\cdot - y_k) \rightharpoonup 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Thus

$$\lim_{k \rightarrow \infty} \left[\int_{\mathbb{R}^N} (-\Delta)^{s/2} u(\cdot - y_k) (-\Delta)^{s/2} v + u(\cdot - y_k) v dx \right] = 0, \quad \forall v \in C_0^\infty(\mathbb{R}^N),$$

where we used in the second term that $\text{supp}(u(\cdot - y_k)) \cap \text{supp}(v) = \emptyset$, for k large enough. \square

Next, we complement the discussion made in [14] by establishing an equivalence between the convergence in $L^p(\mathbb{R}^N)$ and $D_{\mathbb{Z}^N}$ -convergence. The proof of Theorem 2.2 follows next by the same argument found in [65, Corollary 3.3].

Proposition 5.2. *Let (u_k) be a bounded sequence in $H^s(\mathbb{R}^N)$. Then $u_k \xrightarrow{D_{\mathbb{Z}^N}} 0$ in $H^s(\mathbb{R}^N)$, if and only if $u_k \rightarrow 0$ in $L^p(\mathbb{R}^N)$, for all $2 < p < 2_s^*$.*

Proof. The first part is proved in [14, Theorem 2.4]. Thus, let us suppose that $u_k \rightarrow 0$ in $L^p(\mathbb{R}^N)$, $2 < p < 2_s^*$. Take a arbitrary sequence (g_{y_k}) in $D_{\mathbb{Z}^N}$ and let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Using identity (4.5) we have

$$\left| \int_{\mathbb{R}^N} (-\Delta)^{s/2} (g_{y_k}^* u_k) (-\Delta)^{s/2} \varphi \, dx \right| \leq \left(\int_{\mathbb{R}^N} |u_k|^p \, dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |(-\Delta)^s \varphi(\cdot - y_k)|^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}}.$$

Thus, using Hölder inequality again in the L^2 term of the inner product of $H^s(\mathbb{R}^N)$, we conclude that $g_{y_k}^* u_k \rightarrow 0$ in $H^s(\mathbb{R}^N)$. \square

Proof of Theorem 2.2 completed. We prove by applying Theorem E. In fact, let (g_{y_k}) in $D_{\mathbb{Z}^N}$ such that $g_{y_k} \not\rightarrow 0$ in $H^s(\mathbb{R}^N)$. By Lemma 5.1, $y_k \rightarrow y$, up to subsequence, and by [22, Lemma 5.2] $g_{y_k} \rightarrow g_y$. Thus, in view of Proposition D, $(H^s(\mathbb{R}^N), D_{\mathbb{Z}^N})$ is a dislocation space. Assertions (2.6) and (2.8) follows by Lemma 5.1 and Proposition 5.2 respectively. \square

6. VARIATIONAL SETTINGS

This section is devoted to develop the basic background needed in order to apply our variational arguments. We start by establishing the space of functions where the solutions lies.

Proposition 6.1. *Suppose that $V(x) \in L_{\text{loc}}^1(\mathbb{R}^N)$ and satisfies (V_2) , then $H_V^s(\mathbb{R}^N)$ is a Hilbert space continuously embedded in $H^s(\mathbb{R}^N)$. If $V(x)$ satisfies (V_1^*) , then the norm $\|\cdot\|_V$ is equivalent to the standard norm of $\mathcal{D}^{s,2}(\mathbb{R}^N)$.*

Proof. Let us prove first that there exists a positive constant C such that

$$C[\varphi]_s^2 \leq \|\varphi\|_V^2, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \quad (6.1)$$

In fact, on the contrary, there would exist a sequence (φ_n) in $C_0^\infty(\mathbb{R}^N)$, such that

$$[\varphi_n]_s^2 > n \|\varphi_n\|_V^2, \quad \forall n \in \mathbb{N}.$$

Taking $v_n = \varphi_n / [\varphi_n]_s$, we have

$$\frac{1}{n} > \|v_n\|_V^2 \quad \text{and} \quad \mathcal{C}_V \|v_n\|_2^2 \leq \|v_n\|_V^2, \quad \forall n \in \mathbb{N},$$

and consequently $\lim_{n \rightarrow \infty} \|v_n\|_V^2 = \lim_{n \rightarrow \infty} \|v_n\|_2^2 = 0$. This leads to a contradiction with the fact that

$$1 - \mathcal{B} \|v_n\|_2^2 \leq \|v_n\|_V^2, \quad \forall n \in \mathbb{N}.$$

Now consider (φ_n) any sequence in $C_0^\infty(\mathbb{R}^N)$. Using inequality (6.1) we have

$$C[\varphi_m - \varphi_n]_s^2 \leq \|\varphi_m - \varphi_n\|_V^2, \quad \forall m \neq n.$$

Consequently,

$$\|\varphi_m - \varphi_n\|^2 \leq \min\{1, C\}^{-1} \left(1 + \frac{1}{\mathcal{C}_V} \right) \|\varphi_m - \varphi_n\|_V^2, \quad \forall m \neq n.$$

Thus $H_V^s(\mathbb{R}^N)$ is well defined. Moreover, Fatou Lemma and embedding (4.3) implies

$$H_V^s(\mathbb{R}^N) \subset \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\},$$

with the continuous embedding $H_V^s(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$.

Assuming condition (V_1^*) , we have

$$[u]_s^2 + \int_{\mathbb{R}^N} V(x)u^2 dx \geq \mathcal{C}_V^* \int_{\mathbb{R}^N} |V(x)|u^2 dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N),$$

from this we derive

$$\begin{aligned} \mathcal{C}_V^*[u]_s^2 &\leq (\mathcal{C}_V^* + 1)[u]_s^2 + \int_{\mathbb{R}^N} (V(x) - \mathcal{C}_V^*|V(x)|)u^2 dx \\ &\leq (\mathcal{C}_V^* + 1)\|u\|_V^2, \quad \forall u \in C_0^\infty(\mathbb{R}^N). \end{aligned}$$

Since $V(x) \leq 0$ a.e. in \mathbb{R}^N , we conclude that the norms $[\cdot]_s$ and $\|\cdot\|_V$ are equivalent in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. \square

Remark 6.2. (i) If $V(x)$ fulfills (V_2) and (V_4) , then $H_V^s(\mathbb{R}^N) = H^s(\mathbb{R}^N)$. Moreover, the norms $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent. Consequently, the path $\lambda_u(t) := u(\cdot/t)$, $t \geq 0$ belongs to $C([0, \infty), H_V^s(\mathbb{R}^N))$ and $u(\cdot - y) \in H_V^s(\mathbb{R}^N)$ for all $u \in H_V^s(\mathbb{R}^N)$ and $y \in \mathbb{R}^N$. Indeed, there is a ball B_{R_1} with center at the origin such that

$$\begin{aligned} \int_{\mathbb{R}^N} V(x)u^2 dx &= \int_{B_{R_1}} V(x)u^2 dx + \int_{\mathbb{R}^N \setminus B_{R_1}} V(x)u^2 dx \\ &\leq \left(\int_{B_{R_1}} |V(x)|^\sigma dx \right)^{1/\sigma} \left(\int_{B_{R_1}} |u|^{2\sigma/(\sigma-1)} dx \right)^{(\sigma-1)/\sigma} \\ &\quad + (V_\infty + 1) \int_{\mathbb{R}^N \setminus B_{R_1}} u^2 dx, \quad \forall u \in H_V^s(\mathbb{R}^N), \end{aligned}$$

where $2 \leq 2\sigma/(\sigma-1) \leq 2_s^*$. So we can apply embedding (4.3) to conclude the desired result. To obtain that the path λ_u belongs to $H_V^s(\mathbb{R}^N)$ we use [22, Lemma 8.3].

(ii) If we assume assumptions (V_1) – (V_2) , then we can replace $H^s(\mathbb{R}^N)$ by $H_V^s(\mathbb{R}^N)$ in Theorem 2.2 and the respectively norms in the assertions (2.5)–(2.8). In fact, condition (V_1) implies that $D_{\mathbb{Z}^N}$ is a group of unitary operators in $H_V^s(\mathbb{R}^N)$.

Lemma 6.3. *Suppose that $f(x, t)$ satisfies (f_1) and either (f_2) – (f_3) or (f_4) . If $a(x) = V(x) - b(x) \in L_{\text{loc}}^1(\mathbb{R}^N)$ fulfills (V_2) and (V_3) , then the functional I possess the mountain pass geometry. Precisely,*

- (i) $I(0) = 0$;
- (ii) There exists $r, b > 0$ such that $I(u) \geq b$, whenever $\|u\|_V = r$;
- (iii) There is $e \in H_V^s(\mathbb{R}^N)$ with $\|e\|_V > r$ and $I(e) < 0$;

In particular $0 < c(I) < \infty$.

Proof. Let $\xi_R \in C_0^\infty(\mathbb{R})$, $R > 0$, such that $0 \leq \xi_R(t) \leq t_0$ and

$$\xi_R(t) = \begin{cases} t_0, & \text{if } |t| \leq R, \\ 0, & \text{if } |t| > R + 1. \end{cases}$$

Setting $v(x) := \xi_R(|x - x_0|)$, we have $v \in H_V^s(\mathbb{R}^N)$ and by assumption (f₃) we have

$$\begin{aligned} \int_{\mathbb{R}^N} F(x, v) dx &= \int_{B_R(x_0)} F(x, t_0) dx + \int_{B_{R+1}(x_0) \setminus B_R(x_0)} F(x, v) dx \\ &\geq |B_R| \inf_{B_R(x_0)} F(x, t_0) + |B_{R+1} \setminus B_R| \inf_{(x,t) \in (B_{R+1}(x_0) \setminus B_R(x_0)) \times [0, t_0]} F(x, t) > 0. \end{aligned}$$

First assume that (f₂) holds. Since $b(x) \in L^\beta(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} b(x) u^2 dx \leq \left(\int_{\mathbb{R}^N} |b(x)|^\beta dx \right)^{1/\beta} \left(\int_{\mathbb{R}^N} |u|^{2\beta/(\beta-1)} dx \right)^{(\beta-1)/\beta}, \quad \forall u \in H_V^s(\mathbb{R}^N),$$

with $2 < 2\beta/(\beta-1) < 2_s^*$, by conditions (f₁) and (V₃), for any ε we get that

$$I(u) \geq \left[\frac{1}{2} \left(1 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}} - 2\varepsilon \mathcal{C}_2 \right) - \varepsilon \mathcal{C}_{2_s^*} \|u\|_V^{2_s^*-2} - C_\varepsilon \mathcal{C}_{p_\varepsilon} \|u\|_V^{p_\varepsilon-2} \right] \|u\|_V^2, \quad \forall u \in H_V^s(\mathbb{R}^N), \quad (6.2)$$

where \mathcal{C}_2 , $\mathcal{C}_{2_s^*}$ and $\mathcal{C}_{p_\varepsilon}$ are positive constants provided by the embedding described in Proposition 6.1. This allow us to consider ε in a such way that the first term in the right-hand side of (6.2) is positive, once $\|u\|_V$ is taken small enough. Hence there exists $r > 0$ such that $I(u) > 0$ provided that $\|u\|_V = r$. Since condition (f₂) is equivalent to $d/dt(F(x, t)t^{-\mu}) \geq 0$, for $t > 0$, we have

$$\int_{\mathbb{R}^N} F(x, tv) dx \geq t^\mu \int_{\mathbb{R}^N} F(x, v) dx, \quad \text{whenever } t > 1.$$

Hence, as $t \rightarrow \infty$,

$$I(tv) = \frac{t^2}{2} \|v\|_V^2 - \int_{\mathbb{R}^N} b(x) u^2 dx - \int_{\mathbb{R}^N} F(x, tv) dx \leq \frac{t^2}{2} \|v\|_V^2 - t^\mu \int_{\mathbb{R}^N} F(x, v) dx \rightarrow -\infty,$$

Now suppose that assumption (f₄) holds. By Remark 3.5–(ii) we can argue as above to conclude the existence of $r > 0$ such that $I(u) > 0$ wherever $\|u\|_V < r$. For any given $R > 0$, there exists $t_R > 0$ such that

$$F(x, t) > Rt^2, \quad \forall |t| > t_R.$$

Let be $A(R, t) := \{x \in \mathbb{R}^N : t|v(x)| > t_R\}$, for $t > 0$. We have that

$$\begin{aligned} \int_{\mathbb{R}^N} F(x, tv) dx &= \int_{K_t} F(x, tv) dx + \int_{A(R, t)} F(x, tv) dx \\ &\geq \int_{K_t} F(x, tv) dx + Rt^2 \int_{A(R, t)} v^2 dx, \end{aligned} \quad (6.3)$$

where $K_t = (\mathbb{R}^N \setminus A(R, t)) \cap \text{supp}(v)$. Using Remark 3.5–(ii), for each $t > 0$, we get that

$$|F(x, tv)| \leq C, \quad \text{for a.e. } x \in K_t,$$

where C is a positive constant that does not depend in x and t . Consequently, for any $x \in \text{supp } v$,

$$F(x, tv) \mathcal{X}_{K_t}(x) \rightarrow 0, \quad \text{as } t \rightarrow \infty,$$

where we have used that, for any $x \in \text{supp}(v)$,

$$\mathcal{X}_{\mathbb{R}^N \setminus A(R, t)}(x) \rightarrow \mathcal{X}_{\mathbb{R}^N \setminus \text{supp}(v)}(x) = 0, \quad \text{as } t \rightarrow \infty,$$

where \mathcal{X}_A denotes the characteristic function of the set A . Thus Dominated Convergence Theorem implies that the first integral in the right-hand side of inequality (6.3) goes to zero as t goes to infinity. By the same reason, we also have

$$\lim_{t \rightarrow \infty} \int_{A(R, t)} v^2 dx = \lim_{t \rightarrow \infty} \int_{\mathbb{R}^N} v^2 \mathcal{X}_{A(R, t)} dx = \int_{\mathbb{R}^N} v^2 \mathcal{X}_{\{v \neq 0\}} dx = \int_{\mathbb{R}^N} v^2 dx$$

In particular, there exists a positive number $t_{0,R}$ such that

$$\frac{1}{2} \int_{\mathbb{R}^N} v^2 dx < \int_{A(R,t)} v^2 dx, \quad \forall t > t_{0,R}. \quad (6.4)$$

Replacing (6.4) in (6.3) we obtain that

$$\begin{aligned} I(tv) &= \frac{t^2}{2} \|v\|_V^2 - \frac{t^2}{2} \int_{\mathbb{R}^N} b(x) v^2 dx - \int_{\mathbb{R}^N} F(x, tv) dx \\ &\leq \frac{1}{2} (\|v\|_V^2 - R \|v\|_2^2) t^2 - \int_{K_t} F(x, tv) dx < 0, \quad \text{for } t > t_{0,R}, \end{aligned}$$

provided that R is sufficiently large enough. \square

Remark 6.4. (i) In view of Lemma 6.3, we define the set

$$\Gamma_I^1 = \{ \gamma \in C([0, 1], H_V^s(\mathbb{R}^N)) : \gamma(0) = 0, \|\gamma(1)\|_V > r, I(\gamma(1)) < 0 \},$$

and

$$c_1(I) = \inf_{\gamma \in \Gamma_I} \sup_{t \in [0, 1]} I(\gamma(t)), \quad (6.5)$$

the usual minimax level. Thus have $c_1(I) = c(I)$.

- (ii) When $f(x, t) \equiv f(t)$, the mountain pass geometry can be obtained by replacing condition (f_3) by (f'_3) . In fact, let ξ_R as in the proof of Lemma 6.3 and define $\eta_R(x) = \xi_R(|x|)$. Then, arguing as in [21, Remark 2.8], we have

$$\begin{aligned} \int_{\mathbb{R}^N} F(\eta_R) dx &= \int_{B_R(x_0)} F(t_0) dx + \int_{B_{R+1}(x_0) \setminus B_R(x_0)} F(\eta_R) dx \\ &\geq F(t_0) |B_R| - |B_{R+1} \setminus B_R| \left(\max_{t \in [0, t_0]} |F(t)| \right). \end{aligned}$$

Thus there exists two positive constants C_1 and C_2 such that

$$\int_{\mathbb{R}^N} F(\eta_R) dx \geq C_1 R^N - C_2 R^{N-1} > 0,$$

provided that R is sufficiently enough. The mountain pass geometry now follows as in the proof of Lemma 6.3.

- (iii) Assume that $f(x, t)$ satisfies (f_1) and either (f_2) – (f_3) or (f_4) ; and additionally (f_7) . Suppose also that $a(x)$ and $f(x, t)$ fulfills (V_2) – (V_4) and (f_9) , respectively. Then the limiting functional I_∞ has the mountain pass geometry. In fact, (f_9) together with [22, Lemma 8.1] implies that $\lambda_u(t) := u(\cdot/t)$, $t \geq 0$, is an admissible path for Γ_{I_∞} , where $u \in H^s(\mathbb{R}^N)$ is such that

$$\int_{\mathbb{R}^N} F_\infty(u) - \frac{V_\infty}{2} u^2 dx > 0. \quad (6.6)$$

Using the same argument as in Remark 6.4–(ii) we can see that there exists $\varphi_0 \in C_0^\infty(\mathbb{R}^N)$ satisfying (6.6) and

$$I_\infty(\lambda_{\varphi_0}(t)) = \frac{1}{2} t^{N-2s} [\varphi_0]_s^2 - t^N \left[\int_{\mathbb{R}^N} F_\infty(\varphi_0) - \frac{V_\infty}{2} \varphi_0^2 \right] \rightarrow -\infty, \quad \text{as } t \rightarrow \infty.$$

Moreover, $I(u) > 0$ wherever $\|u\|_V = r$, for $r > 0$ sufficiently small enough (see proof of Lemma 6.3).

- (iv) In addition to the assumptions of Lemma 6.3, assume that $F(x, t) > 0$ for a.e. $x \in \mathbb{R}^N$ and $t \neq 0$. Then, for any $u \in H_V^s(\mathbb{R}^N) \setminus \{0\}$, the path defined by $\zeta(t) = tu$ belongs to Γ_I . In fact, we make the following modification in the proof of Lemma 6.3, replacing v by u and taking into account the same notation. We have that

$$\begin{cases} \int_{\mathbb{R}^N} F(x, tu) \, dx \geq Rt^2 \int_{A(R,t)} u^2 \, dx, \\ \lim_{t \rightarrow \infty} \int_{A(R,t)} u^2 \, dx = \lim_{t \rightarrow \infty} \int_{\mathbb{R}^N} u^2 \mathcal{X}_{A(R,t)} \, dx = \int_{\mathbb{R}^N} u^2 \mathcal{X}_{\{u \neq 0\}} \, dx = \int_{\mathbb{R}^N} u^2 \, dx, \end{cases}$$

which enable us to proceed as in (6.4) and get that

$$\varphi(t) := I(tu) \leq \frac{1}{2} (\|u\|_V^2 - R\|u\|_2^2) t^2 \rightarrow -\infty, \text{ as } t \rightarrow \infty,$$

provided that R is large enough. Moreover, assuming that condition (3.5) holds, we can infer that $\zeta(t)$ has a unique critical point.

As a consequence of the previous result, we can guarantee the existence of bounded Palais-Smale sequence at the mountain pass level $c(I)$.

Proposition 6.5. *Assume that $a(x) \in L_{\text{loc}}^1(\mathbb{R}^N)$ fulfills (V₂)–(V₃) and $f(x, t)$ satisfies either*

- (i) (f₁)–(f₃); or
- (ii) (f₃)–(f₆);

Then there exists a bounded sequence (u_k) such that $I(u_k) \rightarrow c(I)$ and $I'(u_k) \rightarrow 0$ in the dual of $H_V^s(\mathbb{R}^N)$.

Proof. (i) By Lemma 6.3, we may apply the standard Mountain Pass Theorem (see [1, 7]) in order to find a sequence (u_k) in $H_V^s(\mathbb{R}^N)$ such that $I(u_k) \rightarrow c(I)$ and $I'(u_k) \rightarrow 0$. For large k , we have

$$\begin{aligned} c(I) + 1 + \|u_k\|_V &\geq I(u_k) - \frac{1}{\mu} I'(u_k) \cdot u_k \\ &= \left(\frac{1}{2} - \frac{1}{\mu} \right) \left(1 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}} \right) \|u_k\|_V^2 - \int_{\mathbb{R}^N} F(x, u_k) - \frac{1}{\mu} f(x, u_k) u_k \, dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \left(1 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}} \right) \|u_k\|_V^2, \end{aligned}$$

which implies that (u_k) is bounded in $H_V^s(\mathbb{R}^N)$.

(ii) The proof follows by the same argument found in [15, Lemma 2.5] and [19, Lemma 4.1]. By Lemma 6.3, we can apply a variant of the Mountain Pass Theorem (see [10, 50]), to obtain the existence of a Cerami sequence (u_k) for I at the level $c(I)$, more precisely,

$$I(u_k) \rightarrow c(I) \text{ and } (1 + \|u_k\|_V) \|I'(u_k)\|_* \rightarrow 0,$$

where $\|\cdot\|_*$ denote the usual norm of the dual of $H_V^s(\mathbb{R}^N)$. We claim that (u_k) is bounded in $H_V^s(\mathbb{R}^N)$. Assume by contradiction that, up to subsequence, $\|u_k\|_V \rightarrow \infty$. Define the sequence

$$v_k = \frac{u_k}{\|u_k\|_V}.$$

We have that

$$\lim_{k \rightarrow \infty} \left[1 - \int_{\mathbb{R}^N} \frac{f(x, u_k)}{\|u_k\|_V} v_k \, dx - \frac{1}{\|u_k\|_V^2} \int_{\mathbb{R}^N} b(x) u_k^2 \, dx \right] = \lim_{k \rightarrow \infty} \left[\frac{1}{\|u_k\|_V^2} I'(u_k) \cdot u_k \right] = 0.$$

The idea is to use indirect arguments and prove that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \frac{f(x, u_k)}{\|u_k\|_V} v_k \, dx = 0,$$

which, by assumption (V₃), leads to the following contradiction,

$$1 = \lim_{k \rightarrow \infty} \frac{1}{\|u_k\|_V^2} \int_{\mathbb{R}^N} b(x) u_k^2 \, dx < \frac{1}{2}. \quad (6.7)$$

For $0 \leq a < b \leq \infty$, defining

$$\Omega_k(a, b) = \{x \in \mathbb{R}^N : a \leq |u_k(x)| \leq b\},$$

we are going to prove that for any given $0 < \varepsilon < 1$, there exists k_ε and real numbers $a_\varepsilon, b_\varepsilon$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{f(x, u_k)}{\|u_k\|_V} v_k \, dx &= \int_{\Omega_k(0, a_\varepsilon)} \frac{f(x, u_k)}{\|u_k\|_V} v_k \, dx \\ &+ \int_{\Omega_k(a_\varepsilon, b_\varepsilon)} \frac{f(x, u_k)}{\|u_k\|_V} v_k \, dx + \int_{\Omega_k(b_\varepsilon, \infty)} \frac{f(x, u_k)}{\|u_k\|_V} v_k \, dx < \varepsilon, \quad \forall k > k_\varepsilon. \end{aligned} \quad (6.8)$$

In order to do that, we first make some estimates involving $\mathcal{F}(x, t)$. Define

$$g(r) = \inf \{ \mathcal{F}(x, t) : x \in \mathbb{R}^N, |t| > r \},$$

which is positive and goes to infinity as $r \rightarrow \infty$. Indeed, thanks to assumptions (f₅) and (f₆), we have

$$a_0 \mathcal{F}(x, t) \geq \left| \frac{f(x, t)}{t} \right|^{p_0} > \left| 2 \frac{F(x, t)}{t^2} \right|^{p_0}, \quad \forall |t| > R_0.$$

Consequently, by condition (f₄), we obtain that $\mathcal{F}(x, t) \rightarrow \infty$, as $|t| \rightarrow \infty$, uniformly in x . Due to assumption (f₅), we also can define the positive number

$$m_a^b = \inf \left\{ \frac{\mathcal{F}(x, t)}{t^2} : x \in \mathbb{R}^N, a \leq |t| \leq b \right\}.$$

Using these notations, we see that there exists k_0 such that

$$\begin{aligned} c(I) + 1 &\geq I(u_k) - \frac{1}{2} I'(u_k) \cdot u_k \\ &= \int_{\Omega_k(0, a)} \mathcal{F}(x, u_k) \, dx + \int_{\Omega_k(a, b)} \mathcal{F}(x, u_k) \, dx + \int_{\Omega_k(b, \infty)} \mathcal{F}(x, u_k) \, dx \\ &\geq \int_{\Omega_k(0, a)} \mathcal{F}(x, u_k) \, dx + m_a^b \int_{\Omega_k(a, b)} u_k^2 \, dx + g(b) |\Omega_k(b, \infty)|, \quad \forall k > k_0. \end{aligned} \quad (6.9)$$

Inequality (6.9) implies

$$\lim_{b \rightarrow \infty} |\Omega_k(b, \infty)| = 0, \quad \text{uniformly in } k > k_0.$$

Moreover, fixed $2 < q \leq 2_s^*$, we have

$$\int_{\Omega_k(a, b)} |v_k|^q \, dx \leq \left(\int_{\Omega_k(a, b)} |v_k|^{2_s^*} \, dx \right)^{q/2_s^*} |\Omega_k(a, b)|^{(2_s^* - q)/2_s^*},$$

in particular,

$$\lim_{b \rightarrow \infty} \int_{\Omega_k(a, b)} |v_k|^q \, dx = 0, \quad \text{uniformly in } k > k_0. \quad (6.10)$$

On the hand, it follows that

$$\int_{\Omega_k(a,b)} v_k^2 dx = \frac{1}{\|u_k\|_V^2} \int_{\Omega_k(a,b)} u_k^2 dx \leq \left(\frac{1}{\|u_k\|_V^2} \right) \left(\frac{1}{(c(I) + 1)m_a^b} \right) \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (6.11)$$

We now pass to prove the estimate (6.8). By condition (f₄), there exists $a_\varepsilon > 0$ such that

$$|f(x, t)| < \varepsilon |t|, \quad \text{a.e. } x \in \mathbb{R}^N, \text{ provided that } |t| < a_\varepsilon.$$

Thus, using (6.11) we have

$$\int_{\Omega_k(0, a_\varepsilon)} \frac{f(x, u_k)}{\|u_k\|_V} v_k dx \leq \int_{\Omega_k(0, a_\varepsilon) \cap \{|u_k| > 0\}} \frac{f(x, u_k)}{|u_k|} v_k^2 dx < \varepsilon/3, \quad \forall k > k_\varepsilon^{(1)}.$$

Taking $2q_0 := 2p_0/(p_0 - 1)$ and using assumption (f₆) we have that

$$\begin{aligned} \int_{\Omega_k(b_\varepsilon, \infty)} \frac{f(x, u_k)}{\|u_k\|_V} v_k dx &\leq \int_{\Omega_k(b_\varepsilon, \infty)} \frac{f(x, u_k)}{|u_k|} v_k^2 dx \\ &\leq (a_0(c(I) + 1))^{1/p_0} \left(\int_{\Omega_k(b_\varepsilon, \infty)} |v_k|^{2q_0} dx \right)^{1/q_0} < \varepsilon/3, \quad \forall k > k_\varepsilon^{(2)}, \end{aligned}$$

where b_ε and $k_\varepsilon^{(2)} > k_0$ are taken from convergence (6.10). Finally, using condition (f₄) we get that

$$|f(x, u_k)| \leq C_\varepsilon |u_k|, \quad \text{a.e. } x \in \Omega_k(a_\varepsilon, b_\varepsilon),$$

and some positive constant C_ε that does not depends on k and x . Thus,

$$\int_{\Omega_k(a_\varepsilon, b_\varepsilon)} \frac{f(x, u_k)}{\|u_k\|_V} v_k dx \leq \int_{\Omega_k(a_\varepsilon, b_\varepsilon)} \frac{f(x, u_k)}{|u_k|} v_k^2 dx \leq C_\varepsilon \int_{\Omega_k(a_\varepsilon, b_\varepsilon)} v_k^2 dx < \varepsilon/3, \quad \forall k > k_\varepsilon^{(3)},$$

where $k_\varepsilon^{(3)} > k_0$ is obtained from (6.11). The contradiction from (6.7) and (6.8) follows by taking $k_\varepsilon \geq \{k_\varepsilon^{(1)}, k_\varepsilon^{(2)}, k_\varepsilon^{(3)}\}$. \square

6.1. Behavior of weak decomposition convergence under nonlinearities. We now pass to describe the limit of the profile decomposition (Theorems A and 2.2) for bounded sequences of the associated functional.

Proposition 6.6. *Suppose that $f(x, t)$ satisfies (f₁), $a(x) \equiv V(x) \in L_{\text{loc}}^1(\mathbb{R}^N)$ and (V₂). Let (u_k) be a bounded sequence in $H_V^s(\mathbb{R}^N)$ such that $u_k \rightarrow u$ in $L^p(\mathbb{R}^N)$, for some $p \in (2, 2_s^*)$, then*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} f(x, u_k) u_k dx = \int_{\mathbb{R}^N} f(x, u) u dx, \quad (6.12)$$

up to subsequence. Moreover, if (v_k) is a bounded sequence in $H_V^s(\mathbb{R}^N)$ with $u_k - v_k \rightarrow 0$ in $L^p(\mathbb{R}^N)$, for some $2 < p < 2_s^$, then, up to subsequence,*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(x, u_k) - F(x, v_k) dx = 0. \quad (6.13)$$

Proof. First observe that $u_k \rightarrow u$ in $L^q(\mathbb{R}^N)$ for all $q \in (2, 2_s^*)$. In fact, this follows by a interpolation inequality, if $q < p$ then

$$\|u_k - u\|_q \leq \|u_k - u\|_2^\theta \|u_k - u\|_p^{1-\theta}$$

where $1/q = \theta/2 + (1 - \theta)/p$, and if $q > p$ then

$$\|u_k - u\|_q \leq \|u_k - u\|_p^\theta \|u_k - u\|_{2_s^*}^{1-\theta}$$

for $1/q = \theta/p + (1 - \theta)/2_s^*$. On the other hand, by (4.4) and Proposition 6.1, $u \in H_V^s(\mathbb{R}^N)$ and

$$u_k(x) \rightarrow u(x) \text{ as } k \rightarrow \infty, \text{ a.e. } x \in \mathbb{R}^N \text{ and } |u_k(x)|, |u(x)| \leq h_\varepsilon(x) \text{ a.e. } x \in \mathbb{R}^N, \quad k \in \mathbb{N},$$

for some $h_\varepsilon \in L^{p_\varepsilon}(\mathbb{R}^N)$. Now note that

$$\int_{\mathbb{R}^N} |f(x, u_k)u_k - f(x, u)u| dx \leq \int_{\mathbb{R}^N} |f(x, u_k)(u_k - u)| dx + \int_{\mathbb{R}^N} |(f(x, u_k) - f(x, u))u| dx.$$

The first integral can be estimated by Hölder inequality as follows

$$\int_{\mathbb{R}^N} |f(x, u_k)(u_k - u)| dx \leq \varepsilon \left(\|u_k\|_2 \|u_k - u\|_2 + \|u_k\|_{2_s^*}^{2_s^*-1} \|u_k - u\|_{2_s^*} \right) + C_\varepsilon \|u_k\|_{p_\varepsilon}^{p_\varepsilon-1} \|u_k - u\|_{p_\varepsilon}.$$

For the second one, consider

$$X_k^\varepsilon := \left\{ x \in \mathbb{R}^N : \varepsilon(|u_k(x)| + |u_k(x)|^{2_s^*-1}) \leq C_\varepsilon |u_k(x)|^{p_\varepsilon-1} \right\}$$

and

$$X^\varepsilon := \left\{ x \in \mathbb{R}^N : \varepsilon(|u(x)| + |u(x)|^{2_s^*-1}) \leq C_\varepsilon |u(x)|^{p_\varepsilon-1} \right\}.$$

Thus

$$\int_{X_k^\varepsilon} |(f(x, u_k) - f(x, u))u| dx = \int_{\mathbb{R}^N} |(f(x, u_k) - f(x, u))u| \mathcal{X}_{X_k^\varepsilon} dx,$$

where \mathcal{X}_Z denotes the characteristic function of the set Z in \mathbb{R}^N . Since $\mathcal{X}_{X_k^\varepsilon}(x) \rightarrow \mathcal{X}_{X^\varepsilon}(x)$ in \mathbb{R}^N and

$$|(f(x, u_k) - f(x, u))u \mathcal{X}_{X_k^\varepsilon}| \leq 2C_\varepsilon h_\varepsilon^{p_\varepsilon} \in L^1(\mathbb{R}^N),$$

we may apply the Dominated Convergence Theorem to conclude

$$\lim_{k \rightarrow \infty} \int_{X_k^\varepsilon} |(f(x, u_k) - f(x, u))u| dx = 0.$$

On the other way,

$$\limsup_{k \rightarrow \infty} \int_{\mathbb{R}^N \setminus X_k^\varepsilon} |(f(x, u_k) - f(x, u))u| dx \leq C\varepsilon.$$

where C is a positive constant that does not depend in ε and k . Since ε is arbitrary, (6.6) holds.

Now, let us prove (6.13). Choose $(\bar{u}_k), (\bar{v}_k)$ in $C_0^\infty(\mathbb{R}^N)$ such that

$$\lim_{k \rightarrow \infty} \|\bar{u}_k - u_k\|_V = \lim_{k \rightarrow \infty} \|\bar{v}_k - v_k\|_V = 0.$$

Thus it suffices to prove that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} [F(x, \bar{u}_k) - F(x, \bar{v}_k)] dx = 0. \quad (6.14)$$

Consider $E := (C_0(\mathbb{R}^N), \|\cdot\|_{p_\varepsilon})$ and the functional $\beta : E \rightarrow \mathbb{R}$, given by $\beta(u) = \int_{\mathbb{R}^N} F(x, u) dx$ with Gateaux derivative

$$\beta'_G(u) \cdot v = \int_{\mathbb{R}^N} f(x, u)v dx.$$

Thus, we may apply the Mean Value Theorem to get

$$|\beta(u) - \beta(v)| \leq \sup_{w \in E, w \in [u, v]} \|\beta'_G(w)\| \|u - v\|_{p_\varepsilon}, \quad \forall u, v \in E, \quad (6.15)$$

where $[u, v] = \{tu + (1 - t)v : t \in [0, 1]\}$. Since $(u_k), (v_k), (\bar{u}_k)$ and (\bar{v}_k) belongs to a bounded set B in $H_V^s(\mathbb{R}^N)$, we also have, by the continuous embedding $H_V^s(\mathbb{R}^N) \hookrightarrow L^{p_\varepsilon}(\mathbb{R}^N)$, that $B \cap E$ is bounded in E . Consequently β'_G is bounded in $B \cap E$, which allows us to take $u = \bar{u}_k$ and $v = \bar{v}_k$ in (6.15) to conclude the convergence (6.14). \square

Our next result can be seen as the nonlocal counterpart of [65, Lemma 5.1] and a generalization of the well known Brezis-Lieb Lemma [6].

Proposition 6.7. *Assume that $f(x, t)$ satisfies (f₁) and (f₇). Let (u_k) in $H^s(\mathbb{R}^N)$ be a bounded sequence and $(w^{(n)})_{n \in \mathbb{N}_0}$ in $H^s(\mathbb{R}^N)$, given by the Theorem 2.2. Then*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(x, u_k) \, dx = \int_{\mathbb{R}^N} F(x, w^{(1)}) \, dx + \sum_{n \in \mathbb{N}_0, n > 1} \int_{\mathbb{R}^N} F_{\mathcal{P}}(x, w^{(n)}) \, dx.$$

Proof. By the Proposition 6.6 the functional

$$\Phi(u) := \int_{\mathbb{R}^N} F(x, u) \, dx, \quad u \in H^s(\mathbb{R}^N),$$

is uniformly continuous in bounded sets of $L^p(\mathbb{R}^N)$, for any $2 < p < 2_s^*$, consequently, by assertions (2.7) and (2.8), we have that

$$\lim_{k \rightarrow \infty} \left[\Phi(u_k) - \Phi \left(\sum_{n \in \mathbb{N}_0} w^{(n)}(\cdot - y_k^{(n)}) \right) \right] = 0.$$

The uniform convergence in (2.8) allows us to reduce to the case where $\mathbb{N}_0 = \{1, \dots, M\}$. Thus taking

$$\Phi_{\mathcal{P}}(u) := \int_{\mathbb{R}^N} F_{\mathcal{P}}(x, u) \, dx, \quad u \in H^s(\mathbb{R}^N),$$

it follows from (f₇) and Dominated Convergence Theorem that

$$\lim_{k \rightarrow \infty} \left[\sum_{n \in \mathbb{N}_0} \Phi \left(w^{(n)}(\cdot - y_k^{(n)}) \right) - \Phi(w^{(1)}) - \sum_{n \in \mathbb{N}_0, n > 1} \Phi_{\mathcal{P}}(w^{(n)}) \right] = 0.$$

It remains to prove that

$$\lim_{k \rightarrow \infty} \left[\Phi \left(\sum_{n \in \mathbb{N}_0} w^{(n)}(\cdot - y_k^{(n)}) \right) - \sum_{n \in \mathbb{N}_0} \Phi \left(w^{(n)}(\cdot - y_k^{(n)}) \right) \right] = 0. \quad (6.16)$$

Since Φ is locally Lipschitz in bounded sets of $H^s(\mathbb{R}^N)$, using a density argument, we can assume without loss of generality that $w^{(n)} \in C_0^\infty(\mathbb{R}^N)$, for $n = 1, \dots, M$. Consequently, from (2.6),

$$\text{supp}(w^{(n)}(\cdot - y_k^{(n)})) \cap \text{supp}(w^{(m)}(\cdot - y_k^{(m)})) = \emptyset, \quad \text{for } m \neq n \text{ and } k \text{ large enough,}$$

which implies that, for k large enough,

$$\begin{aligned} \int_{\mathbb{R}^N} F \left(x, \sum_{n \in \mathbb{N}_0} w^{(n)}(x - y_k^{(n)}) \right) \, dx &= \int_{\bigcup_{n=1}^M \text{supp}(w^{(n)}(\cdot - y_k^{(n)}))} F \left(x, \sum_{m=1}^M w^{(m)}(\cdot - y_k^{(m)}) \right) \, dx \\ &= \sum_{n=1}^M \int_{\text{supp}(w^{(n)})} F(x + y_k^{(n)}, w^{(n)}) \, dx, \end{aligned}$$

from this, (6.16) follows immediately. \square

From Proposition 6.7 it is evident the following result.

Corollary 6.8. *Let (u_k) in $H^s(\mathbb{R}^N)$ be a bounded sequence and $(w^{(n)})_{n \in \mathbb{N}_0}$ in $H^s(\mathbb{R}^N)$, given by Theorem 2.2. If $f(x, t)$ is 1-periodic in each x_i , $i = 1, \dots, N$ and satisfies (f_1) ,*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(x, u_k) dx = \sum_{n \in \mathbb{N}_0} \int_{\mathbb{R}^N} F(x, w^{(n)}) dx. \quad (6.17)$$

Corollary 6.9. *Let $u_k \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ and $F(x, t)$ as in Corollary 6.8 then, up to subsequence,*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} [F(u_k) - F(u - u_k) - F(u)] dx = 0.$$

Proof. Since $w^{(1)} = u$, following the proof of Proposition 6.7, we obtain

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(u_k - u) dx = \sum_{n \in \mathbb{N}_*, n > 1} \int_{\mathbb{R}^N} F(w^{(n)}) dx. \quad (6.18)$$

Taking the difference between (6.17) and (6.18) we get the desired convergence. \square

To treat the case where the nonlinearity has critical growth we use the next result.

Proposition F. [22, Proposition 7.1] *Assume that $0 < s < \min\{1, N/2\}$ and that $f(x, t)$ satisfies (f_1^*) – (f_3^*) . Let $(u_k) \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a bounded sequence and $(w^{(n)})_{n \in \mathbb{N}_*}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $n \in \mathbb{N}_*$, given in Theorem A. Then*

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(x, u_k) dx &= \int_{\mathbb{R}^N} F(x, w^{(1)}) dx \\ &+ \sum_{n \in \mathbb{N}_0, n > 1} \int_{\mathbb{R}^N} F_0(w^{(n)}) dx + \sum_{n \in \mathbb{N}_+} \int_{\mathbb{R}^N} F_+(w^{(n)}) dx + \sum_{n \in \mathbb{N}_-} \int_{\mathbb{R}^N} F_-(w^{(n)}) dx. \end{aligned} \quad (6.19)$$

We also need the following result, that can be understood as an generalization of Fatou Lemma, or alternatively, that the functional $u \mapsto \int_{\mathbb{R}^N} V(x)u^2 dx$ is sequentially weakly lower semicontinuous with respect to the profile decomposition of Theorem 2.2. Moreover, it is a complement to Proposition 6.7.

Proposition 6.10. *Suppose that $a(x) \equiv V(x) \geq 0$ and that (V_2) holds true. Let (u_k) be a bounded sequence in $H^s(\mathbb{R}^N)$ and $(w^{(n)})_{n \in \mathbb{N}_0}$ given in Theorem 2.2.*

(i) *If (V_1) holds, we have*

$$\liminf_{k \rightarrow \infty} \int_{\mathbb{R}^N} V(x)u_k^2 dx \geq \sum_{n \in \mathbb{N}_0} \int_{\mathbb{R}^N} V(x)|w^{(n)}|^2 dx.$$

(ii) *Under (V_4) we obtain,*

$$\liminf_{k \rightarrow \infty} \int_{\mathbb{R}^N} V(x)u_k^2 dx \geq \int_{\mathbb{R}^N} V(x)|w^{(1)}|^2 dx + \sum_{n \in \mathbb{N}_0, n > 1} \int_{\mathbb{R}^N} V_\infty|w^{(n)}|^2 dx.$$

Proof. We prove only the second inequality, the first one follows by a similar argument. It suffices to prove that

$$\begin{aligned} \int_{\mathbb{R}^N} V(x)u_k^2 dx &= \int_{\mathbb{R}^N} \left| |V(x)|^{1/2}(u_k - w^{(1)}) - |V_\infty|^{1/2} \sum_{n=2}^m w^{(n)}(\cdot - y_k^{(n)}) \right|^2 dx \\ &+ \int_{\mathbb{R}^N} V(x)|w^{(1)}|^2 dx + \sum_{n=2}^m \int_{\mathbb{R}^N} V_\infty|w^{(n)}|^2 dx + o(1), \quad \forall m. \end{aligned} \quad (6.20)$$

where with the notation $a_k = o(b_k)$ we mean that $a_k/b_k \rightarrow 0$. To this end, we proceed as in the proof of the iterated Brezis-Lieb Lemma [14, Proposition 6.7], thus the proof of (6.20) is made by induction. We start by checking that (6.20) holds for $m = 2$. In fact, by Proposition 6.1 it is clear that, up to subsequence, the classical Brezis-Lieb Lemma [6] and assertion (2.6) implies that

$$\int_{\mathbb{R}^N} V(x) u_k^2 dx = \int_{\mathbb{R}^N} V(x) |w^{(1)}|^2 dx + \int_{\mathbb{R}^N} V(x) |u_k - w^{(1)}|^2 dx + o(1), \quad (6.21)$$

consequently and by the same reason,

$$\begin{aligned} \int_{\mathbb{R}^N} V(x) |u_k - w^{(1)}|^2 dx &= \\ &= \int_{\mathbb{R}^N} V(x + y_k^{(2)}) |u_k(\cdot + y_k^{(2)}) - w^{(1)}(\cdot + y_k^{(2)})|^2 dx \\ &+ \int_{\mathbb{R}^N} \left| |V(x + y_k^{(2)})|^{1/2} \left(u_k(\cdot + y_k^{(2)}) - w^{(1)}(\cdot + y_k^{(2)}) \right) - |V_\infty w^{(2)}|^{1/2} \right|^2 dx \\ &+ \int_{\mathbb{R}^N} V_\infty |w^{(2)}|^2 dx + o(1). \end{aligned} \quad (6.22)$$

Replacing identity (6.22) in (6.21) we obtain (6.20) for $m = 2$. We shall now prove that (6.20) holds for $m + 1$ provided that it is true for m . Indeed, arguing as above,

$$\begin{aligned} \int_{\mathbb{R}^N} \left| |V(x)|^{1/2} (u_k - w^{(1)}) - V_\infty^{1/2} \sum_{n=2}^m w^{(n)}(\cdot - y_k^{(n)}) \right|^2 dx &- \int_{\mathbb{R}^N} V_\infty |w^{(m+1)}|^2 dx \\ &= \int_{\mathbb{R}^N} \left| |V(x)|^{1/2} (u_k - w^{(1)}) - V_\infty^{1/2} \sum_{n=2}^{m+1} w^{(n)}(\cdot - y_k^{(n)}) \right|^2 dx + o(1). \end{aligned} \quad (6.23)$$

Applying the induction hypothesis in (6.23) we obtain (6.20). \square

6.2. Pohozaev Identity. We finish the section by proving the aforementioned Pohozaev type identity. The proof follows the same arguments used in [22, Section 4] with some appropriated modifications. It complements some well known results in the present literature, namely: [11, Theorem 2.3], [12, Proposition 4.1], [22, Proposition 4.3] and [49, Theorem 1.1].

Proposition 6.11. *Suppose that $f(x, t) \equiv f(t) \in C^1(\mathbb{R})$ and $a(x) \in C^1(\mathbb{R}^N \setminus \mathcal{O})$, where \mathcal{O} is a finite set. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a weak solution of (\mathcal{P}_s) such that $f(u)/(1 + |u|)$ belongs to $L_{\text{loc}}^{N/2s}(\mathbb{R}^N)$. If $F(u)$, $f(u)u$, $a(x)u^2$ and $\langle \nabla a(x), x \rangle u^2$ belongs to $L^1(\mathbb{R}^N)$, then $u \in C^1(\mathbb{R}^N \setminus \mathcal{O})$ and*

$$\frac{N - 2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u| dx + \frac{N}{2} \int_{\mathbb{R}^N} a(x) u^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \langle \nabla a(x), x \rangle u^2 dx = N \int_{\mathbb{R}^N} F(u) dx.$$

Proof. We first prove the local regularity of u . To do that, we consider $x_0 \in \mathbb{R}^N \setminus \mathcal{O}$, and observe that $\bar{u} = u(\cdot + x_0)$ is a weak solution of

$$(-\Delta)^s \bar{u} + \bar{a}(x) \bar{u} = f(\bar{u}) \text{ in } \mathbb{R}^N,$$

where $\bar{a}(x) = a(x + x_0)$. Taking r small enough, the ball B_r^N (see (4.8)) does not contains any point of discontinuity of $\bar{a}(x)$ and so

$$\frac{|g(\bar{u})|}{1 + |\bar{u}|} \in L^{N/2s}(B_r^N), \quad \text{where } g(\bar{u}) := f(\bar{u}) - \bar{a}(x) \bar{u}.$$

This enable us to proceed as in [22, Proposition 4.2], to conclude that $u \in L^p(B_r^N)$, for all $p \geq 1$. Moreover, since

$$g(\bar{u}) = f(\bar{u}) - \bar{a}(x)\bar{u} = \left[\frac{f(\bar{u})}{1 + |\bar{u}|} \operatorname{sgn}(\bar{u}) - \bar{a}(x) \right] \bar{u} + \frac{f(\bar{u})}{1 + |\bar{u}|},$$

we may apply the regularity results of [33] (see also [22, Proposition 4.1]) to conclude that there exists $0 < y_0, r_0 < r$ with $B_r^N \times [0, y_0] \subset B_{r_0}^+$, and $\alpha \in (0, 1)$, such that

$$\bar{w}, \nabla_x \bar{w}, y^{1-2s} \bar{w}_y \in C^{0,\alpha}(B_{r_0}^N \times [0, y_0]),$$

where \bar{w} is the s -harmonic extension of \bar{u} and $\nabla_x \bar{w} = (\bar{w}_{x_1}, \dots, \bar{w}_{x_N})$. In particular, since x_0 is arbitrary,

$$w, \nabla_x w, y^{1-2s} w_y \in C(B_r^N \setminus \mathcal{O} \times [0, y_0]), \quad \forall r, y_0 > 0. \quad (6.24)$$

Consider now $\xi \in C_0^\infty(\mathbb{R} : [0, 1])$ such that

$$\xi(t) = \begin{cases} 1, & \text{if } |t| \leq 1 \\ 0, & \text{if } |t| \geq 2 \end{cases} \quad \text{and} \quad |\xi'(t)| \leq C \quad \forall t \in \mathbb{R},$$

for some $C > 0$. Let $\mathcal{O} = \{x^{(1)}, \dots, x^{(l)}\}$, and $z^{(i)} = (x^{(i)}, 0)$, $i = 1, \dots, l$. For each $n = 1, \dots$, define $\xi_n : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ by

$$\xi_n(z) = \begin{cases} \xi(|z|^2/n^2), & \text{if } |z - z^{(i)}|^2 > 2/n^2, \\ 1 - \xi(n^2|z - z^{(i)}|^2), & \text{if } |z - z^{(i)}|^2 \leq 2/n^2. \end{cases}$$

Then, for n large enough, $\xi_n \in C_0^\infty(\mathbb{R}^N)$ and verifies

$$|z| |\nabla \xi_n(z)| \leq C \quad \forall z \in \mathbb{R}^{N+1}, \quad (6.25)$$

for some $C > 0$. Now observe that

$$\begin{aligned} & \operatorname{div}(y^{1-2s} \nabla w) \langle z, \nabla w \rangle \xi_n \\ &= \operatorname{div} \left[y^{1-2s} \xi_n \left(\langle z, \nabla w \rangle \nabla w - \frac{|\nabla w|^2}{2} z \right) \right] + \frac{N-2s}{2} y^{1-2s} |\nabla w|^2 \xi_n \\ & \quad + y^{1-2s} \frac{|\nabla w|^2}{2} \langle z, \nabla \xi_n \rangle - y^{1-2s} \langle \nabla w, z \rangle \langle \nabla w, \nabla \xi_n \rangle. \end{aligned} \quad (6.26)$$

Given $\delta > 0$ we set the notations

$$\begin{cases} B_{n,\delta} = \{z = (x, y) \in \mathbb{R}_+^{N+1} : |z|^2 < 2n^2, y > \delta\} \\ F_{n,\delta}^1 = \{z = (x, y) \in \mathbb{R}_+^{N+1} : |z|^2 < 2n^2, y = \delta\} \\ F_{n,\delta}^2 = \{z = (x, y) \in \mathbb{R}_+^{N+1} : |x|^2 + y^2 = 2n^2, y > \delta\} \end{cases}$$

Hence $\partial B_{n,\delta} = F_{n,\delta}^1 \cup F_{n,\delta}^2$. Let $\eta(z) = (0, \dots, -1)$ be the unit outward normal vector of $B_{n,\delta}$ over $F_{n,\delta}^1$, by identity (6.26) and the Divergence Theorem we get

$$\begin{aligned} 0 &= \int_{B_{n,\delta}} \operatorname{div}(y^{1-2s} \nabla w) \langle z, \nabla w \rangle \xi_n \, dx dy \\ &= \int_{F_{n,\delta}^1} y^{1-2s} \xi_n \left(\langle z, \nabla w \rangle \langle \nabla w, \eta \rangle - \frac{|\nabla w|^2}{2} \langle z, \eta \rangle \right) dx dy + \theta_{n,\delta} \\ &= \int_{F_{n,\delta}^1} \xi_n \langle x, \nabla_x w \rangle (-y^{1-2s} w_y) \, dx - \int_{F_{n,\delta}^1} y^{1-2s} \xi_n |w_y|^2 y \, dx + \int_{F_{n,\delta}^1} y^{1-2s} \xi_n \frac{|\nabla w|^2}{2} y \, dx + \theta_{n,\delta} \\ &= I_{n,\delta}^1 + I_{n,\delta}^2 + I_{n,\delta}^3 + \theta_{n,\delta}, \end{aligned}$$

where

$$\theta_{n,\delta} = \int_{B_{n,\delta}} \frac{N-2s}{2} y^{1-2s} |\nabla w|^2 \xi_n + y^{1-2s} \frac{|\nabla w|^2}{2} \langle z, \nabla \xi_n \rangle - y^{1-2s} \langle \nabla w, z \rangle \langle \nabla w, \nabla \xi_n \rangle \, dx dy.$$

Using the same arguments as in [24, proof of Theorem 3.7] we deduce that there exists a sequence $\delta_k \rightarrow 0$ such that

$$I_{n,\delta_k}^2 + I_{n,\delta_k}^3 \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Some computations leads to

$$\begin{aligned} &\xi_n(x, 0) \langle x, \nabla u \rangle (f(u) - a(x)u) \\ &= \operatorname{div} \left[\xi_n(x, 0) \left(F(u) - \frac{1}{2} a(x) u^2 \right) x \right] - \langle \nabla \xi_n(x, 0), x \rangle F(u) \\ &\quad - N \xi_n(x, 0) F(u) + \frac{1}{2} \langle \nabla \xi_n(x, 0), x \rangle a(x) u^2 \\ &\quad + \frac{1}{2} \xi_n(x, 0) \langle \nabla a(x), x \rangle u^2 + \frac{N}{2} \xi_n(x, 0) a(x) u^2. \end{aligned}$$

Thus, by condition (6.24) and the Divergence Theorem we have

$$\begin{aligned} \lim_{k \rightarrow \infty} I_{n,\delta_k}^1 &= \kappa_s \int_{B_{\sqrt{2}n}^N} \xi_n(x, 0) \langle x, \nabla u \rangle (f(u) - a(x)u) \, dx \\ &= -\kappa_s \int_{B_{\sqrt{2}n}^N} \langle \nabla \xi_n(x, 0), x \rangle F(u) + N \xi_n(x, 0) F(u) - \frac{1}{2} \langle \nabla \xi_n(x, 0), x \rangle a(x) u^2 \, dx \\ &\quad + \frac{\kappa_s}{2} \int_{B_{\sqrt{2}n}^N} \xi_n(x, 0) \langle \nabla a(x), x \rangle u^2 + \frac{N}{2} \xi_n(x, 0) a(x) u^2 \, dx. \end{aligned}$$

Summing up, we get

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} (I_{n,\delta_k}^1 + I_{n,\delta_k}^2 + I_{n,\delta_k}^3 + \theta_{n,\delta_k}) \\ &= -\kappa_s \int_{B_{\sqrt{2}n}^N} \langle \nabla \xi_n, x \rangle F(u) + N \xi_n F(u) \, dx \\ &\quad + \kappa_s \int_{B_{\sqrt{2}n}^N} \frac{1}{2} \langle \nabla \xi_n, x \rangle a(x) u^2 - \frac{1}{2} \xi_n \langle \nabla a(x), x \rangle u^2 - \frac{N}{2} \xi_n a(x) u^2 \, dx \\ &\quad + \int_{B_{\sqrt{2}n}} \frac{N-2s}{2} y^{1-2s} |\nabla w|^2 \xi_n + y^{1-2s} \frac{|\nabla w|^2}{2} \langle z, \nabla \xi_n \rangle - y^{1-2s} \langle \nabla w, z \rangle \langle \nabla w, \nabla \xi_n \rangle \, dx dy. \end{aligned}$$

Consequently using condition (6.25) to pass the limit as $n \rightarrow \infty$, we conclude that

$$\begin{aligned} \frac{N-2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u| dx &= \frac{N-2s}{2\kappa_s} \int_{\mathbb{R}^N} y^{1-2s} |\nabla w|^2 dx dy \\ &= N \int_{\mathbb{R}^N} F(u) dx - \frac{N}{2} \int_{\mathbb{R}^N} a(x) u^2 - \frac{1}{2} \langle \nabla a(x), x \rangle u^2 dx, \end{aligned}$$

where in the first equality we used condition (4.7). \square

Remark 6.12. In previous the proof we have applied [33, Theorem 2.15] and for that it was crucial that $a(x)$ is a continuously differential function in $\mathbb{R}^N \setminus \mathcal{O}$.

Corollary 6.13. *Assume condition (f_1) and that $f(x, t) = f(t)$ is a continuously differential function. Moreover, that $a(x) \equiv a_0$, where a_0 is a positive constant. If $u \in H^s(\mathbb{R}^N)$ is a weak solution for (\mathcal{P}_s) , then*

$$\int_{\mathbb{R}^N} F(u) - \frac{a_0}{2} u^2 dx = \frac{N-2s}{2N} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx$$

Corollary 6.14. *Assume (f_1^*) and that $f(x, t) = f(t)$ is a continuously differential function. If $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ is a weak solution for (\mathcal{P}_s) , then*

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 - \lambda |x|^{-2s} u^2 dx = \frac{2N}{N-2s} \int_{\mathbb{R}^N} F(u) dx,$$

where $0 < \lambda < \Lambda_{N,s}$ is given by (1.4).

As a direct consequence of Proposition 6.11, we have the following non-existence results, complementing the discussions made in [25, 48].

Corollary 6.15 (Non-existence results). *Assume that $f(x, t) \equiv f(t) \in C^1(\mathbb{R}^N)$ and either one of the following conditions are satisfied,*

- (i) $a(x) \in C^1(\mathbb{R}^N \setminus \mathcal{O})$, where \mathcal{O} is a finite set, $2sa(x) + \langle \nabla a(x), x \rangle > 0$ for all x in a non-zero measure domain and $2_s^* F(t) \leq f(t)t$, for all $t \in \mathbb{R}$; or
- (ii) $a(x) \in C^1(\mathbb{R}^N \setminus \mathcal{O})$, where \mathcal{O} is a finite set, $a(x) > 0$, $\langle \nabla a(x), x \rangle > 0$ for all x in a non-zero measure domain and there exists $0 < \delta \leq 2$, such that $\delta F(t) \geq f(t)t$, for all $t \in \mathbb{R}$; or
- (iii) $a(x) \equiv a_0$ is a positive constant and there exists $0 \leq \delta \leq 2s/(N-2s)$, in a such way that $2_s^* F(t) \leq f(t)t + \delta a_0 t^2$, for all $t \in \mathbb{R}$;
- (iv) $a(x) \equiv 0$ and there exists $0 < p < 2_s^*$ such that $pF(t) \geq f(t)t$ for all $t \in \mathbb{R}$.

If $u \in H^s(\mathbb{R}^N)$ is a weak solution of Eq. (\mathcal{P}_s) , such that $F(u)$, $f(u)u$, $a(x)u^2$, $\langle \nabla a(x), x \rangle u^2$ belongs to $L^1(\mathbb{R}^N)$ and $f(u)/(1+|u|)$ belongs to $L_{\text{loc}}^{N/2s}(\mathbb{R}^N)$, then $u \equiv 0$.

Proof. (i) Applying Proposition 6.11, we get

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx + \frac{N}{N-2s} \int_{\mathbb{R}^N} a(x) u^2 dx + \frac{1}{N-2s} \int_{\mathbb{R}^N} \langle \nabla a(x), x \rangle u^2 dx \leq \int_{\mathbb{R}^N} f(u) u dx,$$

furthermore using that $I'(u) \cdot u = 0$, we obtain

$$\int_{\mathbb{R}^N} (2sa(x) + \langle \nabla a(x), x \rangle) u^2 dx \leq 0,$$

which leads to $u \equiv 0$.

(ii) Using again Proposition 6.11 we obtain that

$$\frac{N-2s}{2N}\delta \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx + \frac{\delta}{2} \int_{\mathbb{R}^N} a(x)u^2 dx + \frac{\delta}{2N} \int_{\mathbb{R}^N} \langle \nabla a(x), x \rangle u^2 dx \geq \int_{\mathbb{R}^N} f(u)u dx,$$

which implies that

$$\left(1 - \frac{N-2s}{2N}\delta\right) \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx + \left(1 - \frac{\delta}{2}\right) \int_{\mathbb{R}^N} a(x)u^2 dx - \frac{\delta}{2N} \int_{\mathbb{R}^N} \langle \nabla a(x), x \rangle u^2 dx \leq 0,$$

from this we get $u \equiv 0$.

(iii) Once more we can use Proposition 6.11 to get

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx + \frac{N}{N-2s}a_0 \int_{\mathbb{R}^N} u^2 dx \geq \int_{\mathbb{R}^N} f(u)u dx,$$

which yields

$$\left[\frac{N - (1+\delta)(N-2s)}{N-2s} \right] a_0 \int_{\mathbb{R}^N} u^2 dx \leq 0.$$

In particular $u \equiv 0$.

(iv) Proposition 6.11 implies that

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx = 2_s^* \int_{\mathbb{R}^N} F(u) dx \geq \frac{2_s^*}{p} \int_{\mathbb{R}^N} f(u)u dx = \frac{2_s^*}{p} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx,$$

which yields $u \equiv 0$. □

7. PROOF OF THEOREM 3.1

Proof. (i) Here we use the profile decomposition given by Theorem 2.2. This makes our argument easier than the one of [15, Theorem 2.1].

By Proposition 6.5 we know that there exists a bounded sequence (u_k) such that $I(u_k) \rightarrow c(I)$ and $I'(u_k) \rightarrow 0$. Since it is bounded, it has a profile decomposition provided by Theorem 2.2. If we have $w^{(n)} = 0$ for all $n \in \mathbb{N}_0$, then by assertion (2.8), $u_k \rightarrow 0$ in $L^p(\mathbb{R}^N)$, for any $2 < p < 2_s^*$ and by (2.5), we have $u_k \rightarrow 0$ in $H_V^s(\mathbb{R}^N)$, up to subsequence. Consequently, by Proposition 6.6, we have

$$\begin{cases} o(1) + c(I) = I(u_k) = \frac{1}{2}\|u_k\|_V^2 - \int_{\mathbb{R}^N} F(x, u_k) dx = \frac{1}{2}\|u_k\|_V^2 + o(1), \\ o(1) = I'(u_k) \cdot u_k = \|u_k\|_V^2 - \int_{\mathbb{R}^N} f(x, u_k)u_k dx = \|u_k\|_V^2 + o(1), \end{cases} \quad (7.1)$$

which is a contradiction, since $c(I) > 0$. Thus, there must be at least one nonzero $w^{(n)}$. Moreover, we have that each $w^{(n)}$ is a critical point of I . In fact, it is well known that, up to subsequence, we can take $h^{(n)}$ in $L^{\sigma'}(\text{supp}(\varphi))$, $n \in \mathbb{N}_0$, such that

$$|u_k(x + y_k^{(n)})| \leq h^{(n)}(x), \quad \text{a.e. } x \in \text{supp}(\varphi), \quad (7.2)$$

where $\sigma' = \sigma/(\sigma - 1)$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$, which can be done thanks to Proposition 6.1. Thus

$$\begin{cases} |V(x + y_k^{(n)})u_k(x + y_k^{(n)})\varphi(x)| = |V(x)u_k(x + y_k^{(n)})\varphi(x)| \leq h^{(n)}(x)|V(x)\varphi(x)| \in L^1(\text{supp}(\varphi)) \\ V(x + y_k^{(n)})u_k(x + y_k^{(n)})\varphi(x) = V(x)u_k(x + y_k^{(n)})\varphi(x) \rightarrow V(x)w^{(n)}(x)\varphi(x), \quad \text{a.e. in } \mathbb{R}^N, \end{cases}$$

which, by the Dominated Convergence Theorem leads to

$$\begin{aligned} \lim_{k \rightarrow \infty} (u_k, \varphi(\cdot - y_k^{(n)}))_V &= \lim_{k \rightarrow \infty} \left[[u_k(\cdot + y_k^{(n)}), \varphi]_s + \int_{\mathbb{R}^N} V(x + y_k^{(n)}) u_k(\cdot + y_k^{(n)}) \varphi(x) dx \right] \\ &= [w^{(n)}, \varphi]_s + \int_{\mathbb{R}^N} V(x) w^{(n)} \varphi dx. \end{aligned}$$

By the same reason and (f₁), up to subsequence we have,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} f(x + y_k^{(n)}, u_k(\cdot + y_k^{(n)})) \varphi dx = \int_{\mathbb{R}^N} f(x, w^{(n)}) \varphi dx.$$

Consequently we may pass the limit in

$$I'(u_k) \cdot \varphi(\cdot - y_k^{(n)}) = (u_k, \varphi(\cdot - y_k^{(n)}))_V - \int_{\mathbb{R}^N} f(x + y_k^{(n)}, u_k(\cdot + y_k^{(n)})) \varphi dx,$$

to conclude that $I'(w^{(n)}) = 0$, for all $n \in \mathbb{N}_0$. In particular, we get that

$$\mathcal{G}_S = \inf \{ I(u) : u \in H_V^s(\mathbb{R}^N) \setminus \{0\}, I'(u) = 0 \},$$

is nonnegative. We are going to prove that \mathcal{G}_S is attained and is positive. Let (u_k) be a minimizing sequence for \mathcal{G}_S , that is $I(u_k) \rightarrow \mathcal{G}_S$ and $I'(u_k) = 0$. Arguing as in Proposition 6.5 we obtain that (u_k) is bounded. Suppose by contradiction and assume that $w^{(n)} = 0$ for all $n \in \mathbb{N}_0$. In this case we actually have that $\mathcal{G}_S > 0$, because on the contrary, if $\mathcal{G}_S = 0$, then using (7.1) we would conclude that $\|u_k\|_V = o(1)$, and at the same time,

$$\|u_k\|_V^2 = \int_{\mathbb{R}^N} f(u_k) u_k dx \leq \varepsilon (C_2 \|u_k\|_V^2 + C_* \|u_k\|_V^{2_s^*}) + C_\varepsilon \|u_k\|_V^{p_\varepsilon},$$

where C_2 , $C_{2_s^*}$ and C_{p_ε} are positive constant obtained by applying the embedding described in Proposition 6.1. In particular,

$$(1 - \varepsilon C_2) \leq \varepsilon C_{2_s^*} \|u_k\|_V^{2_s^* - 2} + C_{p_\varepsilon} \|u_k\|_V^{p_\varepsilon - 2}, \quad \forall k \in \mathbb{N},$$

which, by taking ε small enough, would lead to a contradiction with the fact that $\|u_k\|_V = o(1)$. In view of that, in any case, we can argue as above to conclude that there must be a nonzero $w^{(n_0)}$ that is a critical point of I . We know from (2.5) that $u_k(x + y_k^{(n_0)}) \rightarrow w^{(n_0)}(x)$ a.e. in \mathbb{R}^N , up to subsequence, which allows us to apply Fatou Lemma to get

$$\begin{aligned} \mathcal{G}_S &= \lim_{k \rightarrow \infty} I(u_k) = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \mathcal{F}(x, u_k(\cdot + y_k^{(n_0)})) dx \\ &= \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^N} \mathcal{F}(x, u_k(\cdot + y_k^{(n_0)})) dx \geq \int_{\mathbb{R}^N} \mathcal{F}(x, w^{(n_0)}) dx = I(w^{(n_0)}), \end{aligned}$$

where we have used (f₂) or (f₅) to ensure that $\mathcal{F}(x, u_k(\cdot + y_k^{(n_0)})) = \mathcal{F}(x, u_k) \geq 0$ a.e. in \mathbb{R}^N . Thus, once again using (f₂) or (f₅), we can see that $\mathcal{G}_S = I(w^{(n_0)}) > 0$.

(ii) From Proposition 6.1, the norm

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 - \lambda |x|^{-2s} u^2 dx, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N), \quad 0 < \lambda < \Lambda_{N,s},$$

is equivalent with respect to the norm $[\cdot]_s$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Let (u_k) be a minimizing sequence for \mathcal{I}_λ , and for each k , let u_k^* be the symmetric radial decreasing rearrangement of u_k (see [34] for

more details). Applying the fractional Polya-Szegő inequality (see [4, Theorem 3]), for each k , we have that

$$\begin{cases} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_k^*(x) - u_k^*(y)|^2}{|x - y|^{N+2s}} dx dy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} dx dy, \\ \int_{\mathbb{R}^N} F(u_k^*) dx = \int_{\mathbb{R}^N} F(u_k) dx. \end{cases}$$

Thus $(u_k^*) \subset \mathcal{D}_{\text{rad}}^{s,2}(\mathbb{R}^N)$ and is also a minimizing sequence for (3.4). Now observe that $\|\cdot\|_\lambda$ is invariant with respect to the action of dilations given in Theorem A, more precisely,

$$\|u\|_\lambda^2 = \left\| \gamma^{\frac{N-2s}{2}} u(\gamma^j \cdot) \right\|_\lambda^2, \quad \forall \gamma > 1, u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \text{ and } j \in \mathbb{Z},$$

and satisfies the homogeneity property,

$$\|u(\cdot/\delta)\|_\lambda^2 = \delta^{N-2s} \|u\|_\lambda^2, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N), \delta > 0.$$

In view of Remark 2.1 and Corollary 6.14, we may proceed, using exactly the same argument, as in the proof of [22, Theorem 3.4], replacing $[\cdot]_s$ by $\|\cdot\|_\lambda$. \square

Remark 7.1. (i) In the context of the proof of Theorem 3.1–(i), if we assume in addition that $f(x, t)$ satisfies (3.5), then $\mathcal{G}_S = c(I) = I(w^{(n_0)})$ and $w^{(n_0)}$ is nonnegative. Indeed the truncation given in Remark 4.1 satisfies the assumptions of Theorem 3.1–(i), and we can apply the same argument there, to conclude that the ground state $w^{(n_0)}$ is nonnegative. Furthermore, Remark 6.4–(iv) guarantees that the path $\zeta(t) = tw^{(n_0)}$, $t \geq 0$, belongs to Γ_I and $c(I) \leq I(w^{(n_0)})$. On the other hand, considering (u_k) a minimizing sequence of \mathcal{G}_S , by Corollary 6.8, Remark 6.2–(ii) and estimate (2.7), up to subsequence, we have

$$\mathcal{G}_S = \lim_{k \rightarrow \infty} \left[\frac{1}{2} \|u_k\|_V^2 - \int_{\mathbb{R}^N} F(x, u_k) dx \right] \geq \sum_{n \in \mathbb{N}_0} I(w^{(n)}).$$

Consequently, using (f₂) or (f₅) to guarantee that each $I(w^{(n)})$ is nonnegative, we conclude that $c(I) = \mathcal{G}_S$.

- (ii) If we consider the infimum (3.4) defined over $\mathcal{D}_{\text{rad}}^{s,2}(\mathbb{R}^N)$, by Remark 2.1 we can obtain concentration-compactness of the minimizing sequences as described in [22, Theorem 3.4]. More precisely, for any minimizing sequence (u_k) of (3.4), there exists a sequence (j_k) in \mathbb{Z} such that the sequence $(\gamma^{-\frac{N-2s}{2}j_k} u_k(\gamma^{-j_k} \cdot))$ contains a convergent subsequence in $\mathcal{D}_{\text{rad}}^{s,2}(\mathbb{R}^N)$, whose the limit is a minimizer of (3.4) in $\mathcal{D}_{\text{rad}}^{s,2}(\mathbb{R}^N)$.
- (iii) In the context of the proof of Theorem 3.1 (ii), assume that $F(t) \geq 0$ for all $t \geq 0$. Since $\|u_k\|_\lambda \leq \|u_k\|_\lambda$, without loss of generality we can assume that each u_k is nonnegative. In this case, the obtained minimizer for (3.4) is nonnegative.

8. PROOF OF THEOREM 3.2

Proof. As mentioned, we prove Theorem 3.2 by using the Nehari manifold method (see [60]). For convenience of the reader the proof will be divided into several steps.

Step 1. For each $u \in H_V^s \setminus \{0\}$ there exists a unique $\tau(u) > 0$ such that $\tau(u)u \in \mathcal{N}$ and $\max_{t \geq 0} I(tu) = I(\tau(u)u)$. In particular $\mathcal{N} \neq \emptyset$.

To see that the function $h_u(t) = I(tu)$, $t > 0$, has a maximum point t_u , we proceed in a similar way as in the Remark 6.4–(iv). Moreover, $h'(t_u) = 0$, if and only if $t_u u$ belongs to \mathcal{N} and

$$\|u\|_V^2 - \int_{\mathbb{R}^N} b(x)u^2 dx = \frac{1}{t_u} \int_{\mathbb{R}^N} f(x, t_u u)u dx. \quad (8.1)$$

By condition (3.5) the right-hand side of the above identity occurs at most one point. Thus there is a unique maximum point $\tau(u) = t_u$ for the function $h_u(t)$.

Step 2. The function $\tau : H_V^s \setminus \{0\} \rightarrow (0, \infty)$ is continuous. Thus the map $\eta : H_V^s \setminus \{0\} \rightarrow \mathcal{N}$, defined by $\eta(u) = \tau(u)u$ is continuous and $\eta|_{\mathcal{S}}$ is a homeomorphism of the unit sphere \mathcal{S} of $H_V^s(\mathbb{R}^N)$ in \mathcal{N} .

Assume that $u_n \rightarrow u$ strongly in $H_V^s \setminus \{0\}$. It is well known that the positivity of the primitive $F(x, t)$ together with condition (f₂) implies

$$F(x, t) \geq C_1|t|^\mu - C_2t^2, \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and } \forall t \in \mathbb{R}.$$

Thus, from identity (8.1) we obtain that

$$\|u_n\|_V^2 - \int_{\mathbb{R}^N} b(x)u_n^2 dx \geq C_1|\tau(u_n)|^{\mu-2} \int_{\mathbb{R}^N} |u_n|^\mu dx - C_2\|u_n\|_V^2, \quad \forall n \in \mathbb{N},$$

that is, $(u_n) \subset L^\mu(\mathbb{R}^N)$ with

$$\|u_n\|_V^2 \geq C|\tau(u_n)|^{\mu-2} \int_{\mathbb{R}^N} |u_n|^\mu dx, \quad \forall n \in \mathbb{N}.$$

Moreover, since $u \neq 0$, the sequence (u_n) is bounded from below in the norm $\|\cdot\|_\mu$ by a positive constant. Thus $(\tau(u_n))$ is a bounded sequence. We next prove that any given subsequence of $(\tau(u_n))$ has a convergent subsequence with the same limit $\tau(u)$, from this we obtain the convergence $\tau(u_n) \rightarrow \tau(u)$. It is clear that for a subsequence $\tau(u_n) \rightarrow t_0$. We actually have that t_0 is positive. In fact, using conditions (f₁) and (V₃) in identity (8.1) we get the following estimate,

$$\|u_n\|_V^2 - \int_{\mathbb{R}^N} b(x)u_n^2 dx \leq \varepsilon C \left(\|u_n\|_V^2 + \tau(u_n)^{2_s^*-2} \|u_n\|_V^{2_s^*} \right) + C_\varepsilon \tau(u_n)^{p_\varepsilon-2} \|u_n\|_V^{p_\varepsilon},$$

for all $n \in \mathbb{N}$. From which, we obtain

$$\left(1 - \varepsilon C_2 - \frac{\|b(x)\|_\beta}{C_V^{(\beta)}} \right) \|u_n\|_V^2 \leq \varepsilon C_{2_s^*} \tau(u_n)^{2_s^*-2} \|u_n\|_V^{2_s^*} + C_\varepsilon C_{p_\varepsilon} \tau(u_n)^{p_\varepsilon-2} \|u_n\|_V^{p_\varepsilon}, \quad (8.2)$$

for all $n \in \mathbb{N}$, which implies $t_0 > 0$, by taking ε small enough. Thus we may apply the Dominated Convergence Theorem in (8.1) to conclude that $t_0 = \tau(u)$ and the continuity of the function τ . Using (8.1) to compute $\tau(u/\|u\|_V)$ we obtain that

$$\|u\|_V^2 - \int_{\mathbb{R}^N} b(x)u^2 dx = \frac{1}{\frac{\tau(u/\|u\|_V)}{\|u\|_V}} \int_{\mathbb{R}^N} f\left(x, \frac{\tau(u/\|u\|_V)}{\|u\|_V}\right) u dx,$$

which by uniqueness gives $\tau(u/\|u\|_V) = \tau(u)u$. Consequently the inverse of η is the retraction map given by $\varrho : \mathcal{N} \rightarrow \mathcal{S}$, $\varrho(u) = u/\|u\|_V$.

Step 3. \mathcal{N} is away from the origin, that is, there is a ball $B_{R_{\mathcal{N}}}$ with center at the origin such that $\mathcal{N} \subset \mathbb{R}^N \setminus B_{R_{\mathcal{N}}}(0)$.

Indeed, estimate (8.2) implies that

$$1 - \varepsilon C_2 - \frac{\|b(x)\|_\beta}{C_V^{(\beta)}} \leq \varepsilon C_{2_s^*} \|u\|_V^{2_s^*-2} + C_\varepsilon C_{p_\varepsilon} \|u\|_V^{p_\varepsilon-2}, \quad \forall u \in \mathcal{N}.$$

Taking ε small enough we see that $\|u\| \geq C$, for all $u \in \mathcal{N}$.

Step 4. For all $\zeta \in \Gamma_I$ we have that $\zeta([0, \infty)) \cap \mathcal{N} \neq \emptyset$.

Let us suppose that this assertion is false, that is, there exists $\zeta_0 \in \Gamma_I$ which does not intercepts \mathcal{N} in any point. Let $t_0 > 0$ such that $I(\zeta_0(t_0)) < 0$ and $\zeta_0(t) \neq 0$, for all $(0, t_0]$. We prove now that $\tau(\zeta(t)) > 1$ for all $t \in (0, t_0]$. In fact, by continuity, there is a positive number δ such that

$\|\zeta_0(t)\| < R_{\mathcal{N}}$, for all $t \in [0, \delta]$. At the same time, we have that $\|\tau(\zeta_0(t))\zeta_0(t)\|_V > R_{\mathcal{N}}$, which implies $\tau(\zeta_0(t)) > 1$, for all $t \in (0, \delta]$. The continuity of $\tau(t)$ and the fact that $\zeta_0(t) \notin \mathcal{N}$, for all t , allow us to choose $\delta = t_0$. On the other hand, by conditions (f_2) and (3.5) , we have that

$$\begin{aligned} h_{\zeta(t_0)}(t) &\geq \frac{t^2}{2} \left[\|\zeta_0(t_0)\|_V^2 - \int_{\mathbb{R}^N} b(x)|\zeta_0(t_0)|^2 dx - \frac{2}{\mu} \int_{\mathbb{R}^N} \frac{f(x, t\zeta_0(t_0))}{t\zeta_0(t_0)} |\zeta_0(t_0)|^2 dx \right] \\ &> \frac{t^2}{2} \left[\int_{\mathbb{R}^N} \frac{f(x, \tau(\zeta_0(t_0))\zeta_0(t_0))}{\tau(\zeta_0(t_0))\zeta_0(t_0)} |\zeta_0(t_0)|^2 - \frac{f(x, t\zeta_0(t_0))}{t\zeta_0(t_0)} |\zeta_0(t_0)|^2 dx \right] \\ &> 0, \quad \forall t \in (0, \tau(\zeta(t_0))]. \end{aligned}$$

In particular, $0 < h_{\zeta(t_0)}(1) = I(\zeta_0(t_0))$, which is a contradiction with the choice of $\zeta_0(t_0)$.

Step 5. $c_{\mathcal{N}}(I) = \bar{c}(I)$.

In fact, since $\eta|_S$ is a homeomorphism, we have

$$\bar{c}(I) = \inf_{u \in H_V^s \setminus \{0\}} I(\tau(u)u) = \inf_{u \in S} I(\tau(u)u) = c_{\mathcal{N}}(I).$$

Step 6. $\bar{c}(I) = c(I)$.

Given $u \in H_V^s \setminus \{0\}$, define the path $\zeta(t) = tt_0u$, where $t_0 > 0$ is chosen in such way that $I(t_0u) < 0$. Then, by Remark 6.4–(iv), it is easy to see that $\zeta \in \Gamma_I$ and

$$\max_{t \geq 0} I(tu) = \max_{t \geq 0} I(\zeta(t)) \geq c(I).$$

Consequently $c(I) \leq \bar{c}(I)$. On the other hand, given $\zeta \in \Gamma_I$, we know about the existence of t_0 such that $\zeta(t_0)$ belongs to \mathcal{N} . Thus,

$$\max_{t \geq 0} I(\zeta(t)) \geq I(\zeta(t_0)) \geq c_{\mathcal{N}}(I) = \bar{c}(I).$$

Since $\zeta \in \Gamma_I$ is arbitrary, we can conclude $c(I) \geq \bar{c}(I)$. \square

Remark 8.1. In view of Remark 4.1, if $b(x) \equiv 0$, then the radial ground state solution u obtained in Theorem 3.2 can be considered as being nonnegative.

9. PROOF OF THEOREM 3.3

Before the proof of Theorem 3.3, to complement our discussion, we are going to compare the minimax level of limit functionals $I_{\mathcal{P}}$ and I_{∞} with the minimax level of the energy functional I associated with Eq. (\mathcal{P}_s) . Some arguments used to prove this result of comparison are used in the proof of Theorem 3.3.

Proposition 9.1. *Assume that $f(x, t)$ satisfies either (f_1) – (f_3) , (f_7) or (f_3) – (f_6) , (f_7) . Moreover, suppose that $b(x) \equiv 0$, (V_1) – (V_2) and (f_8) . Then $c(I) \leq c(I_{\mathcal{P}})$. Alternatively, if instead of the last set of hypothesis we assume that $V(x) \geq 0$, $b(x)$ has compact support, (V_2) – (V_4) and (f_9) , then $c(I) \leq c(I_{\infty})$. Moreover, under these conditions, if we assume (\mathcal{H}) , then (f_{10}) and (f'_{10}) holds true respectively for each considered case.*

Proof. (i) Let $u \in H_V^s(\mathbb{R}^N)$ be a nonnegative (see Remark 4.1) nontrivial weak solution for the equation

$$(-\Delta)^s u + V(x)u = f_{\mathcal{P}}(x, u),$$

at the mountain pass level for $I_{\mathcal{P}}$, that is, $I_{\mathcal{P}}(u) = c(I_{\mathcal{P}})$. For each k , we define the path

$$\zeta_k(t) = tu(\cdot - y_k), \quad t \geq 0.$$

where (y_k) is taken such that $|y_k| \rightarrow \infty$. The idea is to prove that

$$c(I) \leq \limsup_{k \rightarrow \infty} \max_{t \geq 0} I(\zeta_k(t)) \leq \max_{t \geq 0} I_{\mathcal{P}}(tu) = c(I_{\mathcal{P}}). \quad (9.1)$$

In fact, taking into account that I and $I_{\mathcal{P}}$ are locally Lipschitz sets of $H_V^s(\mathbb{R}^N)$ (they are C^1 in $H_V^s(\mathbb{R}^N)$) and the following estimate

$$|I(\zeta_k(t)) - I_{\mathcal{P}}(tu)| \leq \int_{\mathbb{R}^N} |F(x + y_k, tu) - F_{\mathcal{P}}(x + y_k, tu)| \, dx,$$

by using a density argument we get that

$$\lim_{k \rightarrow \infty} I(\zeta_k(t)) = I_{\mathcal{P}}(tu), \quad \text{uniformly in compact sets of } \mathbb{R}.$$

Consequently we may proceed as in [22, Proposition 9.1]. First note that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(x + y_k, tu) \, dx = \int_{\mathbb{R}^N} F_{\mathcal{P}}(x, tu) \, dx, \quad \text{for each } t > 0.$$

In particular,

$$\int_{\mathbb{R}^N} F(x + y_k, u) \, dx > 0, \quad \text{for } k \text{ large enough.}$$

Thus, using either (f₁)–(f₃) or (f₃)–(f₆) and the arguments of Remark 6.4–(iv), we see that, for k large enough, ζ_k belongs to Γ_I . As a consequence, there exist $t_k > 0$ such that

$$I(\zeta_k(t_k)) = \max_{t \geq 0} I(\zeta_k(t)) > 0.$$

We claim that (t_k) is bounded. In fact, assume by contradiction that up to subsequence $t_k \rightarrow \infty$. Thus, by the arguments of Remark 6.4–(iv) we get

$$I(\zeta_k(t_k)) = \frac{t_k^2}{2} \|u\|_V^2 - \int_{\mathbb{R}^N} F(x + y_k, t_k u) \, dx \rightarrow -\infty, \quad \text{as } t \rightarrow \infty,$$

which leads to a contradiction with the fact that $I(\zeta_k(t_k)) > 0$ for all k . Therefore, up to subsequence, $t_k \rightarrow t_0$, and thus

$$\lim_{k \rightarrow \infty} \max_{t \geq 0} I(\zeta_k(t)) = I_{\mathcal{P}}(t_0 u),$$

which leads to (9.1).

(ii). The second case is proved in a similar way. Let $w \in H_V^s(\mathbb{R}^N) = H^s(\mathbb{R}^N)$ be a nontrivial weak solution for the equation

$$(-\Delta)^s w + V_{\infty} w = f_{\infty}(w),$$

at the mountain pass level, more precisely, $I_{\infty}(w) = c(I_{\infty})$. For each k , define the path

$$\lambda_k(t) = w \left(\frac{\cdot - y_k}{t} \right), \quad t \geq 0.$$

where (y_k) is chosen in a such way that $|y_k| \rightarrow \infty$. As before, we consider the estimate

$$\begin{aligned} & |I(\lambda_k(t)) - I_{\infty}(w(\cdot/t))| \\ & \leq \frac{t^N}{2} \int_{\mathbb{R}^N} |(V(tx + y_k) - b(tx + y_k)) - V_{\infty}| w^2 \, dx + t^N \int_{\mathbb{R}^N} |F(tx + y_k, w) - F_{\infty}(w)| \, dx, \end{aligned}$$

and the fact that I and I_{∞} are Lipschitz in bounded sets of $H^s(\mathbb{R}^N)$ to obtain, by a density argument, that

$$\lim_{k \rightarrow \infty} I(\lambda_k(t)) = I_{\infty}(w(\cdot/t)), \quad \text{uniformly in compact sets of } \mathbb{R}.$$

We also have that the path λ_k belongs to Γ_I , for k large enough. In fact, assuming the contrary, we would obtain k_0 and a sequence $t_n \rightarrow \infty$ such that $I(\lambda_{k_0}(t_n)) > 0$, for all n . On the other hand, we have that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(t_n x + y_{k_0}, w) - \frac{1}{2} [V(t_n x + y_{k_0}) - b(t_n x + y_{k_0})] w^2 dx = \int_{\mathbb{R}^N} F_\infty(w) - \frac{1}{2} V_\infty w^2 dx,$$

which leads to the contradiction $I(\lambda_{k_0}(t_n)) < 0$, if n large enough. Let $t_k > 0$ such that

$$I(\lambda_k(t_k)) = \max_{t \geq 0} I(\lambda_k(t)) > 0.$$

We claim that (t_k) is a bounded sequence. On the contrary, there is a subsequence (t_{n_k}) that implies in the following contradiction

$$\begin{aligned} 0 &< I(\lambda_k(t_{n_k})) \\ &= \frac{1}{2} t_{n_k}^{N-2s} [w]_s^2 - t_{n_k}^N \left[\int_{\mathbb{R}^N} F(t_{n_k} x + y_k, w) - \frac{1}{2} (V(t_{n_k} x + y_k) - b(t_{n_k} x + y_k) w^2) dx \right] \\ &\rightarrow -\infty, \text{ as } k \rightarrow \infty. \end{aligned}$$

Thus, up to subsequence, $t_k \rightarrow t_0$ and we obtain that

$$\lim_{k \rightarrow \infty} \max_{t \geq 0} I(\lambda_k(t)) = I_\infty(w(\cdot/t_0)).$$

As a consequence we conclude that

$$c(I) \leq \lim_{k \rightarrow \infty} \max_{t \geq 0} I(\lambda_k(t_k)) \leq \max_{t \geq 0} I_\infty(w(\cdot/t)) = c(I_\infty),$$

where we have used Corollary 6.13 to conclude that $t = 1$ is the unique critical point of $I_\infty(w(\cdot/t))$.

Now assume (\mathcal{H}) . Considering the above discussion, for each case respectively, we have

$$\begin{cases} c(I) \leq \max_{t \geq 0} I(\zeta_k(t)) = I(t_k u(\cdot - y_k)) < I_{\mathcal{P}}(t_k u) \leq \max_{t \geq 0} I_{\mathcal{P}}(t_k u) = c(I_{\mathcal{P}}), \\ c(I) \leq \max_{t \geq 0} I(\lambda_k(t)) = I(u((\cdot - y_k)/t_k)) < I_\infty(u(\cdot/t_k)) \leq \max_{t \geq 0} I_\infty(u(\cdot/t_k)) = c(I_\infty), \end{cases}$$

for k large enough. \square

In order to prove our existence result without the compactness condition (f_{10}) and (f'_{10}) , we use a similar argument as made in [15, proof of Theorem 1.2]. Thus we need the following result, which states that the existence of a critical point of I is guaranteed whenever the minimax level (3.2) is attained (see Remark 6.4–(i)).

Theorem G. [38, Theorem 2.3] *Let E be a real Banach space. Suppose that $I \in C^1(E)$ satisfies*

- (i) $I(0) = 0$;
- (ii) *There exists $r, b > 0$ such that $I(u) \geq b$, whenever $\|u\| = r$;*
- (iii) *There is $e \in E$ with $\|e\|_V > r$ and $I(e) < 0$;*

Let

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \|\gamma(1)\| > r, I(\gamma(1)) < 0\}.$$

If there exists $\gamma_0 \in \Gamma$ such that

$$c = \max_{t \in [0,1]} I(\gamma_0(t)),$$

then I possess a nontrivial critical point $u \in \gamma_0([0,1])$ such that $I(u) = c$.

Proof of Theorem 3.3 completed. In view of Lemma 6.3 and Proposition 6.5 there exists a bounded sequence (u_k) such that $I(u_k) \rightarrow c(I)$ and $I'(u_k) \rightarrow 0$, for both considered cases of this theorem. Let be the sequences $(w^{(n)})$ and $(y_k^{(n)})$ provided by the Theorem 2.2 for the sequence (u_k) . The underlying main idea to proof the concentration-compactness of Theorem 3.3, follows the same one of [22, Theorem 3.6] which we shall now describe: we prove that $w^{(n)} = 0$ for all $n \geq 2$, which by assertions (2.5), (2.8) and Proposition 6.6 implies that $u_k \rightarrow w^{(1)}$ in $H_V^s(\mathbb{R}^N)$, up to subsequence. In order to prove that, we argue by contradiction and assume the existence of at least one $w^{(n_0)} \neq 0$, $n_0 \geq 2$.

(i) In view of Remark 6.2–(ii), by Proposition 6.7 and estimate (2.7), up to subsequence, we have

$$c(I) = \lim_{k \rightarrow \infty} \left[\frac{1}{2} \|u_k\|_V^2 - \int_{\mathbb{R}^N} F(x, u_k) dx \right] \geq I(w^{(1)}) + \sum_{n \in \mathbb{N}_0, n > 1} I_{\mathcal{P}}(w^{(n)}), \quad (9.2)$$

where each term of the right-hand side of (9.2) is nonnegative. In fact, following as in the proof of Theorem 3.1 we notice that $w^{(1)}$ and $w^{(n)}$, $n \geq 2$, are critical points for I and $I_{\mathcal{P}}$, respectively. In view of that, it is clear that (f₂) or (f₅) implies that $I(w^{(1)}) \geq 0$ and $I_{\mathcal{P}}(w^{(n)}) \geq 0$, $n \geq 2$, respectively. On the other hand, Remark 6.4–(iv) guarantees that the path $\zeta^{(n_0)}(t) = tw^{(n_0)}$ belongs to $\Gamma_{I_{\mathcal{P}}}$ and $c(I_{\mathcal{P}}) < I_{\mathcal{P}}(w^{(n_0)})$. This, together with (9.2) and (f₁₀) leads to a contradiction.

(ii) Following the proof of Theorem 2.2 it is clear that we can replace $\|\cdot\|$ by the equivalent norm $\|\cdot\|_{V_\infty}$ in assertions (2.5)–(2.8). Consequently, by estimate (2.7), Propositions 6.7 and 6.10, up to subsequence, we have

$$\begin{aligned} c(I) &\geq \lim_{k \rightarrow \infty} \left[\frac{1}{2} \|u_k\|_V^2 - \int_{\mathbb{R}^N} b(x) u_k^2 dx - \int_{\mathbb{R}^N} F(x, u_k) dx \right] \\ &\geq I(w^{(1)}) + \sum_{n \in \mathbb{N}_0, n > 1} I_\infty(w^{(n)}). \end{aligned} \quad (9.3)$$

Thus, it suffices to prove that the right-hand side of (9.3) is non-negative and $I_\infty(w^{(n)}) \geq c(I_\infty)$ for all $n \geq 2$. In fact, in this case, we have $c(I) \geq I(w^{(n_0)}) \geq c(I_\infty)$, which leads to a contradiction with (f₁₀). To do this, we prove that $w^{(1)}$ and $w^{(n)}$ are critical points for I and I_∞ respectively, $n \geq 2$. Let φ in $C_0^\infty(\mathbb{R}^N)$ and $h^{(n)} \in L^{2^*-1}(\text{supp}(\varphi))$ as in (7.2). By (V₄) and (2.6), there exists $k_0 = k_0(\varphi)$ such that

$$V(x + y_k^{(n)}) < 1 + V_\infty, \quad \forall k > k_0, \quad x \in \text{supp}(\varphi) \quad \text{and} \quad n \geq 2.$$

Thus,

$$\begin{cases} |V(x + y_k^{(n)}) u_k(x + y_k^{(n)}) \varphi(x)| \leq (\varepsilon + V_\infty) h^{(n)}(x) |\varphi(x)| \in L^1(\text{supp}(\varphi)), \text{ for } k > k_0, \\ V(x + y_k^{(n)}) u_k(x + y_k^{(n)}) \varphi(x) \rightarrow V_\infty w^{(n)}(x) \varphi(x) \quad \text{a.e. in } \mathbb{R}^N, \end{cases}$$

which together with the Dominated Convergence Theorem implies

$$\begin{aligned} \lim_{k \rightarrow \infty} (u_k, \varphi(\cdot - y_k^{(n)}))_V &= \lim_{k \rightarrow \infty} \left[[u_k(\cdot + y_k^{(n)}), \varphi]_s + \int_{\mathbb{R}^N} V(x + y_k^{(n)}) u_k(\cdot + y_k^{(n)}) \varphi(x) dx \right] \\ &= [w^{(n)}, \varphi]_s + \int_{\mathbb{R}^N} V_\infty w^{(n)}(x) \varphi(x) dx. \end{aligned}$$

And for the same reason,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} f(x + y_k^{(n)}, u_k(\cdot + y_k^{(n)})) \varphi dx = \int_{\mathbb{R}^N} f_\infty(w^{(n)}) \varphi dx.$$

Consequently, taking the limit in

$$I'(u_k) \cdot \varphi(\cdot - y_k^{(n)}) = (u_k, \varphi(\cdot - y_k^{(n)}))_V - \int_{\mathbb{R}^N} f(x + y_k^{(n)}, u_k(\cdot + y_k^{(n)})) \varphi \, dx, \quad (9.4)$$

we deduce that $I'(w^{(1)}) = 0$ and $I'_\infty(w^{(n)}) = 0$, $n \geq 2$. Using (f₂) or (f₅) we also get that $I(w^{(1)}) \geq 0$ and $I_\infty(w^{(n)}) \geq 0$, $n \geq 2$. Finally, define the path $\lambda^{(n_0)}(t) = w^{(n_0)}(\cdot/t)$, $t \geq 0$. By Corollary 6.13 we have that

$$I_\infty(\lambda^{(n_0)}(t)) = \frac{1}{2} t^{N-2s} [w^{(n_0)}]_s^2 - t^N \left[\int_{\mathbb{R}^N} F_\infty(w^{(n_0)}) - \frac{V_\infty}{2} |w^{(n_0)}|^2 \, dx \right] \rightarrow -\infty, \text{ as } t \rightarrow \infty,$$

which together with Remark 6.2 implies that $\lambda^{(n_0)}$ belongs to Γ_{I_∞} . Corollary 6.13 also implies that $t = 1$ is the unique critical point of $I_\infty(\lambda^{(n_0)}(t))$. Consequently,

$$c(I_\infty) < \max_{t \geq 0} I_\infty(\lambda^{(n_0)}(t)) = I_\infty(w^{(n_0)}),$$

a contradiction.

(iii) Finally, assume condition (3.6) instead of (f₁₀) and (f'₁₀). Consider the existence of $w^{(n_0)} \neq 0$, $n_0 \in \mathbb{N}_0$, and the paths $\zeta^{(n_0)}$ and $\lambda^{(n_0)}$ as above. Taking account the above discussion, by estimates (9.2) and (9.3), for each case we have

$$\begin{cases} c(I) \leq \max_{t \geq 0} I(\zeta^{(n_0)}(t)) \leq \max_{t \geq 0} I_{\mathcal{P}}(\zeta^{(n_0)}(t)) = I_{\mathcal{P}}(w^{(n_0)}) \leq c(I), \\ c(I) \leq \max_{t \geq 0} I(\lambda^{(n_0)}(t)) \leq \max_{t \geq 0} I_\infty(\lambda^{(n_0)}(t)) = I_\infty(w^{(n_0)}) \leq c(I), \end{cases}$$

where we have used condition (3.6) to ensure that the paths $\zeta^{(n_0)}$ and $\lambda^{(n_0)}$ belongs to Γ_I . Thus, we have that the minimax level $c(I)$ is attained and we can apply Theorem G to obtain the existence of a critical point u for I_λ with $I_\lambda(u) = c(I_\lambda)$. If there is no $w^{(n)} \neq 0$, $n \in \mathbb{N}_0$, (which is the case where strict inequalities occurs) we can argue as above and obtain that $u_k \rightarrow w^{(1)}$, up to subsequence. \square

10. PROOF OF THEOREM 3.4

Proof. The proof will be divided into three steps. Our argument follows the proof of Theorem 3.3 and [13, Theorem 5.2]. We first assume the case where $V(x)$ and $f(x, t)$ satisfies (\mathcal{H}^*) and (\mathcal{H}_0^*).

(i) Arguing in a similar way as in the proof of Lemma 6.3, we see that the functional I_* has the mountain pass geometry, which guarantees the existence of a sequence (u_k) in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ such that $I_*(u_k) \rightarrow c(I_*) > 0$ and $I'_*(u_k) \rightarrow 0$. Let $(w^{(n)})$, $(y_k^{(n)})$, $(j_k^{(n)})$ be the sequences given by Theorem A and define the set

$$\mathbb{N}_\# = \left\{ n \in \mathbb{N}_* \setminus \{1\} : |\gamma^{j_k^{(n)}} y_k^{(n)}| \text{ is bounded} \right\}.$$

Passing to a subsequence and using a diagonal argument if necessary, we may assume that each sequence $(\gamma^{j_k^{(n)}} y_k^{(n)})$, $n \in \mathbb{N}_\#$, with

$$a^{(n)} = \lim_{k \rightarrow \infty} \gamma^{j_k^{(n)}} y_k^{(n)}, \quad n \in \mathbb{N}_\#.$$

(ii) Now we shall prove the following estimate,

$$\limsup_k \|u_k\|_V^2 \geq \|w^{(1)}\|_V^2 + \sum_{n \in \mathbb{N}_* \setminus \mathbb{N}_\#} [w^{(n)}]_s^2 + \sum_{n \in \mathbb{N}_+ \cap \mathbb{N}_\#} \|w^{(n)}\|_{V_+(\cdot + a^{(n)} - a_*)}^2 + \sum_{n \in \mathbb{N}_- \cap \mathbb{N}_\#} \|w^{(n)}\|_{V_-(\cdot + a^{(n)} - a_*)}^2, \quad (10.1)$$

passing in a subsequence of (u_k) if necessary. In order to prove this, first consider $d_k^{(n)}$ the operator

$$d_k^{(n)} u = \gamma^{\frac{N-2s}{2} j_k^{(n)}} u(\gamma^{j_k^{(n)}}(\cdot - y_k^{(n)})), \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N), \quad n \in \mathbb{N}_*.$$

For each $n \in \mathbb{N}_*$, let $(\varphi_j^{(n)})$ in $C_0^\infty(\mathbb{R}^N)$ such that $\varphi_j^{(n)} \rightarrow w^{(n)}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Evaluating

$$\left\| u_k - \sum_{n \in M_*} d_k^{(n)} \varphi_j^{(n)} \right\|_V^2 \geq 0,$$

in a finite subset $M_* = \{1, \dots, M\}$ of \mathbb{N}_* , we have

$$\|u_k\|_V^2 \geq 2 \sum_{n \in M_*} (u_k, d_k^{(n)} \varphi_j^{(n)})_V - \sum_{n \in M_*} \|d_k^{(n)} \varphi_j^{(n)}\|_V^2. \quad (10.2)$$

We are now going to study the limit in inequality (10.2). Let

$$v_k^{(n)} := d_k^{(n)} u_k = \gamma^{-\frac{N-2s}{2} j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}}(\cdot + y_k^{(n)})).$$

Notice that

$$(u_k, d_k^{(n)} \varphi_j^{(n)})_V = [v_k^{(n)}, \varphi_j^{(n)}]_s + \int_{\mathbb{R}^N} \gamma^{-2s j_k^{(n)}} V(\gamma^{-j_k^{(n)}}((x + y_k^{(n)}) + a_*)) v_k^{(n)}(\cdot + a_*) \varphi_j^{(n)}(\cdot + a_*) dx,$$

and

$$\|d_k^{(n)} \varphi_j^{(n)}\|_V^2 = [\varphi_j^{(n)}]_s^2 + \int_{\mathbb{R}^N} \gamma^{-2s j_k^{(n)}} V(\gamma^{-j_k^{(n)}}((x + y_k^{(n)}) + a_*)) |\varphi_j^{(n)}(\cdot + a_*)|^2 dx.$$

Fixed j , we can use condition (V_3^*) to conclude, up to a subsequence that

$$\lim_{k \rightarrow \infty} (u_k, d_k^{(n)} \varphi_j^{(n)})_V = [w^{(n)}, \varphi_j^{(n)}]_s \quad \text{and} \quad \lim_{k \rightarrow \infty} \|d_k^{(n)} \varphi_j^{(n)}\|_V^2 = [\varphi_j^{(n)}]_s^2, \quad (10.3)$$

provided that $n \notin \mathbb{N}_\#$ (this is the case when $n \in \mathbb{N}_0$). Similarly, up to a subsequence, by assumption (V_2^*) we have

$$\lim_{k \rightarrow \infty} (u_k, d_k^{(n)} \varphi_j^{(n)})_V = (w^{(n)}, \varphi_j^{(n)})_{V_\kappa(\cdot + a^{(n)} - a_*)} \quad \text{and} \quad \lim_{k \rightarrow \infty} \|d_k^{(n)} \varphi_j^{(n)}\|_V^2 = \|\varphi_j^{(n)}\|_{V_\kappa(\cdot + a^{(n)} - a_*)}^2, \quad (10.4)$$

where $\kappa = +, -$, whenever $n \in \mathbb{N}_+ \cap \mathbb{N}_\#$ or $\mathbb{N}_- \cap \mathbb{N}_\#$, respectively. Since

$$\mathbb{N}_* \setminus \{1\} = (\mathbb{N}_* \setminus \mathbb{N}_\#) \dot{\cup} [(\mathbb{N}_+ \cap \mathbb{N}_\#) \dot{\cup} (\mathbb{N}_- \cap \mathbb{N}_\#)],$$

up to subsequence, we can apply the limits (10.3) and (10.4) in inequality (10.2) to get

$$\begin{aligned} \limsup_k \|u_k\|_V^2 &\geq \|w^{(1)}\|_V^2 + \sum_{n \in M_* \cap \mathbb{N}_+ \cap \mathbb{N}_\#} 2(w^{(n)}, \varphi_j^{(n)})_{V_+(\cdot+a^{(n)}-a_*)} - \|\varphi_j^{(n)}\|_{V_+(\cdot+a^{(n)}-a_*)}^2 \\ &\quad + \sum_{n \in M_* \cap \mathbb{N}_- \cap \mathbb{N}_\#} 2(w^{(n)}, \varphi_j^{(n)})_{V_-(\cdot+a^{(n)}-a_*)} - \|\varphi_j^{(n)}\|_{V_-(\cdot+a^{(n)}-a_*)}^2 \\ &\quad + \sum_{n \in M_* \setminus \mathbb{N}_\#} 2[w^{(n)}, \varphi_j^{(n)}]_s - [\varphi_j^{(n)}]_s^2. \end{aligned} \quad (10.5)$$

Since the norms $\|\cdot\|_{V_+}$ and $\|\cdot\|_{V_-}$ are equivalent to the norm $[\cdot]_s$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ we can take the limit in j in inequality (10.5) and use the arbitrariness of choice for M to obtain (10.1).

(iii) If $w^{(n)} = 0$ for all $n \geq 2$, then $u_k \rightarrow w^{(1)}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, with $w^{(1)}$ being a critical point of I_* . Let us argue by contradiction and assume the existence of $w^{(n_0)} \neq 0$, with $n_0 \geq 2$. By Proposition F and estimate (10.1), up to subsequence, we have that

$$c(I_*) \geq I_*(w^{(1)}) + \sum_{n \in \mathbb{N}_* \setminus \mathbb{N}_\#} I_0(w^{(n)}) + \sum_{n \in \mathbb{N}_+ \cap \mathbb{N}_\#} I_+^{(n)}(w^{(n)}) + \sum_{n \in \mathbb{N}_- \cap \mathbb{N}_\#} I_-^{(n)}(w^{(n)}), \quad (10.6)$$

where

$$I_\pm^{(n)}(u) = \frac{1}{2} \|u\|_{V_\pm(\cdot+a^{(n)}-a_*)}^2 - \int_{\mathbb{R}^N} F_\pm(u) \, dx \quad \text{and} \quad I_0(u) = \frac{1}{2} [u]_s^2 - \int_{\mathbb{R}^N} F_0(u) \, dx, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

As before, we prove that each $w^{(n)}$ is a critical point for the functionals in the respective index of the sums in (10.6), and as a consequence of (f₂), the right-hand side of (10.6) is non-negative. In the next step we obtain that $c(I_*) < I_\kappa^{(n)}(w^{(n)})$ in the correspondent index, which leads to a contradiction with estimate (10.6). In fact, given φ in $C_0^\infty(\mathbb{R}^N)$, by reasoning as in the proof of (10.1), we get that

$$\lim_{k \rightarrow \infty} (u_k, d_k^{(n)} \varphi)_V = [w^{(n)}, \varphi]_s \quad \text{and} \quad \lim_{k \rightarrow \infty} (u_k, d_k^{(n)} \varphi)_V = (w^{(n)}, \varphi)_{V_\pm(\cdot+a^{(n)}-a_*)},$$

provided that $n \in \mathbb{N}_* \setminus \mathbb{N}_\#$ and $n \in \mathbb{N}_\pm \cap \mathbb{N}_\#$, respectively. Since,

$$\left| \gamma^{-\frac{N+2s}{2} j_k^{(n)}} f\left(\gamma^{-j_k^{(n)}} x + y_k^{(n)}, \gamma^{\frac{N-2s}{2} j_k^{(n)}} t\right) \varphi \right| \leq C |t|^{2s-1}, \quad \forall k, n \text{ and } t,$$

thanks to the Dominated Convergence Theorem, up to a subsequence, we may pass the limit in k in the following identity

$$I'_*(u_k) \cdot (d_k^{(n)} \varphi) = (v_k^{(n)}, \varphi)_V - \int_{\mathbb{R}^N} \gamma^{-\frac{N+2s}{2} j_k^{(n)}} f\left(\gamma^{-j_k^{(n)}} x + y_k^{(n)}, \gamma^{\frac{N-2s}{2} j_k^{(n)}} v_k^{(n)}\right) \varphi \, dx,$$

to conclude that $I'_*(w^{(1)}) = (I_\pm^{(n)})'(w^{(n)}) = I'_0(w^{(n)}) = 0$, in the corresponding index.

(iv) To conclude the proof, we prove now that $c(I_*) < I_\pm^{(n_0)}(w^{(n_0)})$ or $c(I_*) < I_\pm^{(n_0)}(w^{(n_0)})$, where n_0 belongs to $\mathbb{N}_* \setminus \mathbb{N}_\#$ or $\mathbb{N}_\pm \cap \mathbb{N}_\#$ respectively. Define the path

$$\begin{cases} \zeta^{(n_0)}(t) = tw^{(n_0)}, & t \geq 0, \quad \text{if } n_0 \in \mathbb{N}_* \setminus \mathbb{N}_\#. \\ \zeta^{(n_0)}(t) = tw^{(n_0)}(\cdot + a_* - a^{(n)}), & t \geq 0, \quad \text{if } n_0 \in \mathbb{N}_\pm \cap \mathbb{N}_\#. \end{cases}$$

By condition $(\mathcal{H}^*)-(\mathcal{H}_0^*)$ and Remark 6.4-(iv) we have that $\zeta^{(n_0)}$ belongs to Γ_I with

$$\begin{cases} c(I_*) \leq \max_{t \geq 0} I_*(\zeta^{(n_0)}(t)) < I_0(\zeta^{(n_0)}(\bar{t})) \leq \max_{t \geq 0} I_0(\zeta^{(n_0)}(t)) = I_0(w^{(n_0)}), & \text{if } n_0 \in \mathbb{N}_* \setminus \mathbb{N}_\#. \\ c(I_*) \leq \max_{t \geq 0} I_*(\zeta^{(n_0)}(t)) < I_\pm^{(n)}(\zeta^{(n_0)}(\bar{t})) \leq \max_{t \geq 0} I_\pm^{(n)}(\zeta^{(n_0)}(t)) = I_\pm^{(n)}(w^{(n_0)}), & \text{if } n_0 \in \mathbb{N}_\pm \cap \mathbb{N}_\#, \end{cases}$$

where \bar{t} is the maximum of $I_*(\zeta^{(n_0)}(t))$. This together with the estimate (10.6) leads to a aforementioned contradiction.

(v) We now assume only conditions (\mathcal{H}^*) . Arguing in a similar way as above, we get that

$$u_k \rightarrow w^{(1)} \text{ in a subsequence} \quad \text{or} \quad c(I_*) = \max_{t \geq} I_*(\zeta^{(n_0)}(t)).$$

If the minimax level $c(I_*)$ is attained then we apply Proposition G to obtain the existence of a critical point $u \in \zeta^{(n_0)}([0, \infty))$ such that $I_*(u) = c(I_*)$. \square

Acknowledgments. Research supported in part by INCTmat/MCT/Brazil, CNPq and CAPES/Brazil

REFERENCES

- [1] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis* **14** (1973) 349–381. [11](#), [24](#)
- [2] D. Applebaum, Lévy processes—from probability to finance and quantum groups, *Notices Amer. Math. Soc.* **51** (2004) 1336–1347. [2](#)
- [3] D. Applebaum, *Lévy processes and stochastic calculus*, *Cambridge Studies in Advanced Mathematics*, vol. 116, Cambridge University Press, Cambridge, 2nd ed. (2009). [2](#)
- [4] W. Beckner, Sobolev inequalities, the Poisson semigroup, and analysis on the sphere S^n , *Proc. Nat. Acad. Sci. U.S.A.* **89** (1992) 4816–4819. [36](#)
- [5] H. Brezis, How to recognize constant functions. A connection with Sobolev spaces, *Uspekhi Mat. Nauk* **57** (2002) 59–74. [15](#)
- [6] H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88** (1983) 486–490. [28](#), [30](#)
- [7] H. Brezis and L. Nirenberg, Remarks on finding critical points, *Comm. Pure Appl. Math.* **44** (1991) 939–963. [24](#)
- [8] L. Caffarelli, Non-local diffusions, drifts and games, *Nonlinear partial differential equations*, *Abel Symp.*, vol. 7, Springer, Heidelberg (2012) 37–52. [2](#)
- [9] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007) 1245–1260. [4](#)
- [10] G. Cerami, An existence criterion for the critical points on unbounded manifolds, *Istit. Lombardo Accad. Sci. Lett. Rend. A* **112** (1978) 332–336 (1979). [24](#)
- [11] X. Chang, Ground states of some fractional Schrödinger equations on \mathbb{R}^N , *Proc. Edinb. Math. Soc. (2)* **58** (2015) 305–321. [30](#)
- [12] X. Chang and Z.-Q. Wang, Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity, *Nonlinearity* **26** (2013) 479–494. [30](#)
- [13] D. G. Costa, J. M. do Ó and K. Tintarev, Schrödinger equations with critical nonlinearity, singular potential and a ground state, *J. Differential Equations* **249** (2010) 240–252. [3](#), [11](#), [42](#)
- [14] M. Cwikel and K. Tintarev, On interpolation of cocompact imbeddings, *Rev. Mat. Complut.* **26** (2013) 33–55. [4](#), [6](#), [20](#), [30](#)
- [15] R. de Marchi, Schrödinger equations with asymptotically periodic terms, *Proc. Roy. Soc. Edinburgh Sect. A* **145** (2015) 745–757. [2](#), [3](#), [10](#), [11](#), [12](#), [24](#), [34](#), [40](#)
- [16] M. de Souza, J. M. do Ó and T. da Silva, On a class quasilinear Schrödinger equations in \mathbb{R}^n , *Appl. Anal.* **95** (2016) 323–340. [3](#)
- [17] Y. Deng, L. Jin and S. Peng, Solutions of Schrödinger equations with inverse square potential and critical nonlinearity, *J. Differential Equations* **253** (2012) 1376–1398. [3](#)
- [18] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** (2012) 521–573. [2](#), [15](#)
- [19] Y. Ding and C. Lee, Multiple solutions of Schrödinger equations with indefinite linear part and super or asymptotically linear terms, *J. Differential Equations* **222** (2006) 137–163. [11](#), [24](#)
- [20] S. Dipierro, L. Montoro, I. Peral and B. Sciunzi, Qualitative properties of positive solutions to nonlocal critical problems involving the Hardy-Leray potential <http://arxiv.org/abs/1506.07317v1> . [3](#), [11](#)

- [21] S. Dipierro, G. Palatucci and E. Valdinoci, Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian, *Matematiche (Catania)* **68** (2013) 201–216. [23](#)
- [22] J. M. do Ó and D. Ferraz, Concentration-compactness principle for nonlocal scalar field equations with critical growth <https://arxiv.org/abs/1609.06501> (2016). [3](#), [5](#), [9](#), [12](#), [19](#), [20](#), [21](#), [23](#), [29](#), [30](#), [31](#), [36](#), [39](#), [41](#)
- [23] E. B. Fabes, C. E. Kenig and R. P. Serapioni, The local regularity of solutions of degenerate elliptic equations, *Comm. Partial Differential Equations* **7** (1982) 77–116. [16](#)
- [24] M. M. Fall and V. Felli, Unique continuation property and local asymptotics of solutions to fractional elliptic equations, *Comm. Partial Differential Equations* **39** (2014) 354–397. [4](#), [32](#)
- [25] M. M. Fall and T. Weth, Monotonicity and nonexistence results for some fractional elliptic problems in the half-space, *Commun. Contemp. Math.* **18** (2016) 1550012, 25. [33](#)
- [26] V. Felli and A. Pistoia, Existence of blowing-up solutions for a nonlinear elliptic equation with Hardy potential and critical growth, *Comm. Partial Differential Equations* **31** (2006) 21–56. [11](#)
- [27] V. Felli and S. Terracini, Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity, *Comm. Partial Differential Equations* **31** (2006) 469–495. [11](#)
- [28] P. Felmer, A. Quaas and J. Tan, Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A* **142** (2012) 1237–1262. [4](#), [6](#)
- [29] B. Feng, Ground states for the fractional Schrödinger equation, *Electron. J. Differential Equations* (2013) No. 127, 11. [2](#)
- [30] P. Gérard, Description du défaut de compacité de l’injection de Sobolev, *ESAIM Control Optim. Calc. Var.* **3** (1998) 213–233 (electronic). [4](#)
- [31] I. W. Herbst, Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$, *Comm. Math. Phys.* **53** (1977) 285–294. [3](#)
- [32] S. Jaffard, Analysis of the lack of compactness in the critical Sobolev embeddings, *J. Funct. Anal.* **161** (1999) 384–396. [4](#)
- [33] T. Jin, Y. Li and J. Xiong, On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions, *J. Eur. Math. Soc. (JEMS)* **16** (2014) 1111–1171. [4](#), [16](#), [31](#), [33](#)
- [34] S. Kesavan, *Symmetrization & applications*, *Series in Analysis*, vol. 3, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2006). [35](#)
- [35] N. S. Landkof, *Foundations of modern potential theory*, Springer-Verlag, New York-Heidelberg (1972), translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180. [2](#)
- [36] R. Lehrer, L. A. Maia and M. Squassina, Asymptotically linear fractional Schrödinger equations, *Complex Var. Elliptic Equ.* **60** (2015) 529–558. [2](#), [3](#)
- [37] S. Li, Y. Ding and Y. Chen, Concentrating standing waves for the fractional Schrödinger equation with critical nonlinearities, *Bound. Value Probl.* (2015) 2015:240. [3](#)
- [38] H. F. Lins and E. A. B. Silva, Quasilinear asymptotically periodic elliptic equations with critical growth, *Nonlinear Anal.* **71** (2009) 2890–2905. [12](#), [40](#)
- [39] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984) 109–145. [12](#)
- [40] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984) 223–283. [2](#), [12](#)
- [41] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. I, *Rev. Mat. Iberoamericana* **1** (1985) 145–201. [12](#)
- [42] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. II, *Rev. Mat. Iberoamericana* **1** (1985) 45–121. [12](#)
- [43] O. H. Miyagaki, On a class of semilinear elliptic problems in \mathbf{R}^N with critical growth, *Nonlinear Anal.* **29** (1997) 773–781. [3](#)
- [44] A. Nekvinda, Characterization of traces of the weighted Sobolev space $W^{1,p}(\Omega, d_M^\alpha)$ on M , *Czechoslovak Math. J.* **43(118)** (1993) 695–711. [16](#)
- [45] G. Palatucci and A. Pisante, Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces, *Calc. Var. Partial Differential Equations* **50** (2014) 799–829. [4](#), [5](#)
- [46] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, *CBMS Regional Conference Series in Mathematics*, vol. 65, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI (1986). [7](#)

- [47] P. H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.* **43** (1992) 270–291. [3](#), [11](#)
- [48] X. Ros-Oton and J. Serra, Fractional Laplacian: Pohozaev identity and nonexistence results, *C. R. Math. Acad. Sci. Paris* **350** (2012) 505–508. [33](#)
- [49] X. Ros-Oton and J. Serra, The Pohozaev identity for the fractional Laplacian, *Arch. Ration. Mech. Anal.* **213** (2014) 587–628. [30](#)
- [50] M. Schechter, A variation of the mountain pass lemma and applications, *J. London Math. Soc. (2)* **44** (1991) 491–502. [24](#)
- [51] I. Schindler and K. Tintarev, Mountain pass solutions to semilinear problems with critical nonlinearity, *Discrete Contin. Dyn. Syst.* (2007) 912–919. [3](#)
- [52] S. Secchi, Ground state solutions for nonlinear fractional Schrödinger equations in \mathbb{R}^N , *J. Math. Phys.* **54** (2013) 031501, 17. [2](#), [10](#), [11](#), [18](#)
- [53] X. Shang and J. Zhang, Ground states for fractional Schrödinger equations with critical growth, *Nonlinearity* **27** (2014) 187–207. [3](#)
- [54] X. Shang, J. Zhang and Y. Yang, On fractional Schrödinger equation in \mathbb{R}^N with critical growth, *J. Math. Phys.* **54** (2013) 121502, 20. [3](#)
- [55] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.* **60** (2007) 67–112. [2](#), [18](#)
- [56] B. Sirakov, Existence and multiplicity of solutions of semi-linear elliptic equations in \mathbf{R}^N , *Calc. Var. Partial Differential Equations* **11** (2000) 119–142. [3](#)
- [57] D. Smets, Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities, *Trans. Amer. Math. Soc.* **357** (2005) 2909–2938 (electronic). [3](#), [11](#)
- [58] S. Solimini, A note on compactness-type properties with respect to Lorentz norms of bounded subsets of a Sobolev space, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **12** (1995) 319–337. [4](#)
- [59] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, *Math. Z.* **187** (1984) 511–517. [4](#)
- [60] A. Szulkin and T. Weth, The method of Nehari manifold, *Handbook of nonconvex analysis and applications*, Int. Press, Somerville, MA (2010) 597–632. [36](#)
- [61] S. Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, *Adv. Differential Equations* **1** (1996) 241–264. [11](#)
- [62] K. Tintarev, Concentration-compactness principle for mountain pass problems, *Differential & difference equations and applications*, Hindawi Publ. Corp., New York (2006) 1055–1060. [11](#)
- [63] K. Tintarev, Positive solutions of elliptic equations with a critical oscillatory nonlinearity, *Discrete Contin. Dyn. Syst.* (2007) 974–981. [3](#), [12](#)
- [64] K. Tintarev, Concentration compactness at the mountain pass level in semilinear elliptic problems, *NoDEA Nonlinear Differential Equations Appl.* **15** (2008) 581–598. [3](#), [11](#)
- [65] K. Tintarev and K.-H. Fieseler, *Concentration compactness*, Imperial College Press, London (2007), functional-analytic grounds and applications. [3](#), [4](#), [5](#), [6](#), [18](#), [19](#), [20](#), [28](#)
- [66] B. O. Turesson, *Nonlinear potential theory and weighted Sobolev spaces*, *Lecture Notes in Mathematics*, vol. 1736, Springer-Verlag, Berlin (2000). [16](#)
- [67] D. Yafaev, Sharp constants in the Hardy-Rellich inequalities, *J. Funct. Anal.* **168** (1999) 121–144. [3](#)
- [68] H. Zhang, J. Xu and F. Zhang, Existence and multiplicity of solutions for superlinear fractional Schrödinger equations in \mathbb{R}^N , *J. Math. Phys.* **56** (2015) 091502, 13. [2](#), [3](#)

(J.M. do Ó) DEPARTMENT OF MATHEMATICS, FEDERAL UNIVERSITY OF PARAÍBA
 58051-900, JOÃO PESSOA-PB, BRAZIL
 E-mail address: jmbo@pq.cnpq.br

(D. Ferraz) DEPARTMENT OF MATHEMATICS, FEDERAL UNIVERSITY OF PARAÍBA
 58051-900, JOÃO PESSOA-PB, BRAZIL
 E-mail address: diego.ferraz.br@gmail.com